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CHAPTER 8

INTEGRAL EQUATIONS

Integral equations have arrested the interest of mathematicians for over a century (see Elliott 1980). Most integral equations do not have closed-form solutions; however, they can often be discretized and solved on a digital computer. Proof of the existence of the solution to an integral equation by discretization was first presented by Fredholm (1903). But such a discretization procedure was not feasible until the advent of the digital computer, which, in recent decades, created a "boom" in interest concerning integral equations.

When inhomogeneities are piecewise constant in each region, we may solve such problems using the surface integral equation technique. In this technique, the homogeneous-medium Green's functions are found for each region. Then, the field in each region is written in terms of the field due to any sources in the region plus the field due to surface sources at the interfaces between the regions, following Huygens' principle. Next, the boundary conditions at these interfaces are used to set up integral equations known as surface integral equations. From these integral equations, the unknown surface sources at the interfaces can be solved for.

The surface integral equation method is rather popular in a number of applications, because it employs a homogeneous-medium Green's function which is simple in form, and the unknowns are on a surface rather than in a volume. Moreover, the surface integral equation method is not limited to piecewise-constant inhomogeneities. For example, if each region is a layered medium whose Green's function is available as shown in Chapter 7, surface integral equations can be formulated for unknowns at interfaces between such regions.

For a bounded inhomogeneity, an alternative method is to view the scattered field as due to the induced currents flowing in the inhomogeneity. Now, the induced currents are proportional to the total field in the inhomogeneity. But in turn, the total field is the incident field plus the field due to induced currents in the inhomogeneity. So, this concept yields an equation called the volume integral equation from which the unknown field inside the inhomogeneity can be solved for.

In this chapter, the surface integral equations[1] both for scalar and vector fields will be studied first. Then, the volume integral equation will be discussed next.

§8.1 Surface Integral Equations

In an integral equation, the unknown to be sought is embedded in an integral. When the unknown of a linear integral equation is embedded inside the integral only, the integral equation is of the first kind. But when the unknown occurs both inside and outside the integral, the integral equation is of the second kind. An integral equation can be viewed as an operator equation. Thus, the matrix representation of such an operator equation (Chapter 5) can be obtained, and then the unknown is easily solved for with a computer. Later, we shall see how such integral equations with only surface integrals are derived, beginning with the scalar wave equation, followed by the vector wave equation case.

An early form of such a surface integral equation was first derived by Green (1818) through the use of Green's theorem. Since then, his idea has been adapted to solve different problems. Many of these works are listed in the references for this chapter. Poggio and Miller (1973), in addition, provide an extensive reference list for works in this area. The advantage of the surface integral equations is that they reduce the dimensionality of a problem by one. For example, a three-dimensional problem is reduced to a lower dimensional problem involving surface integrals.

§§*8.1.1 Scalar Wave Equation*

Consider a scalar wave equation for a two-region problem as shown in Figure 8.1.1. In region 1, the governing equation for the total field is

$$(\nabla^2 + k_1^2)\,\phi_1(\mathbf{r}) = Q(\mathbf{r}), \tag{8.1.1}$$

while in region 2, it is

$$(\nabla^2 + k_2^2)\,\phi_2(\mathbf{r}) = 0. \tag{8.1.2}$$

Therefore, we can define Green's functions for regions 1 and 2 respectively to satisfy the following equations:

$$(\nabla^2 + k_1^2)\,g_1(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'), \tag{8.1.3}$$

$$(\nabla^2 + k_2^2)\,g_2(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'). \tag{8.1.4}$$

On multiplying Equation (1) by $g_1(\mathbf{r},\mathbf{r}')$ and Equation (3) by $\phi_1(\mathbf{r})$, subtracting the two resultant equations, and integrating over region 1, we have, for

[1] These are sometimes called boundary integral equations.

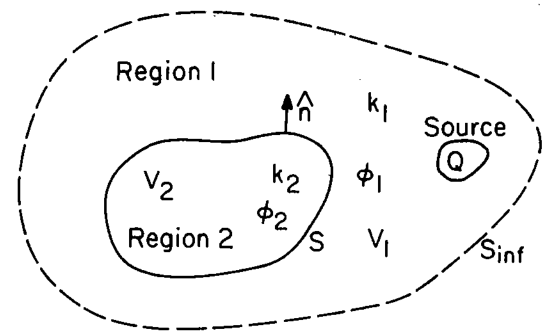


Figure 8.1.1 A two-region problem can be solved with a surface integral equation.

$\mathbf{r}' \in V_1$,

$$\int_{V_1} dV\,[g_1(\mathbf{r},\mathbf{r}')\nabla^2\phi_1(\mathbf{r}) - \phi_1(\mathbf{r})\nabla^2 g_1(\mathbf{r},\mathbf{r}')] = \int_{V_1} dV\,g_1(\mathbf{r},\mathbf{r}')Q(\mathbf{r}) + \phi_1(\mathbf{r}'), \quad \mathbf{r}' \in V_1. \tag{8.1.5}$$

Since $\nabla\cdot(g\nabla\phi - \phi\nabla g) = g\nabla^2\phi - \phi\nabla^2 g$, by applying Gauss' theorem, the volume integral on the left-hand side of (5) becomes a surface integral over the surface bounding V_1. Consequently,[2]

$$-\int_{S+S_{inf}} dS\,\hat{n}\cdot[g_1(\mathbf{r},\mathbf{r}')\nabla\phi_1(\mathbf{r}) - \phi_1(\mathbf{r})\nabla g_1(\mathbf{r},\mathbf{r}')] = -\phi_{inc}(\mathbf{r}') + \phi_1(\mathbf{r}'), \quad \mathbf{r}' \in V_1. \tag{8.1.6}$$

In the above, we have let

$$\phi_{inc}(\mathbf{r}') = -\int_{V_1} dV\,g_1(\mathbf{r},\mathbf{r}')Q(\mathbf{r}), \tag{8.1.7}$$

since it is the incident field generated by the source $Q(\mathbf{r})$. Note that up to this point, $g_1(\mathbf{r},\mathbf{r}')$ is not explicitly specified, and the manipulation up to (6)

[2] The equality of the volume integral on the left-hand side of (5) and the surface integral on the left-hand side of (6) is also known as Green's theorem.

is legitimate as long as $g_1(\mathbf{r},\mathbf{r}')$ is a solution of (3). For example, a possible choice for $g_1(\mathbf{r},\mathbf{r}')$ that satisfies the radiation condition is[3]

$$g_1(\mathbf{r},\mathbf{r}') = \frac{e^{ik_1|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}, \tag{8.1.8}$$

which is the unbounded, homogeneous medium scalar Green's function (see Subsection 1.3.4). In this case, $\phi_{inc}(\mathbf{r})$ is the incident field generated by the source $Q(\mathbf{r})$ in the absence of the scatterer. Moreover, the integral over S_{inf} vanishes when $S_{inf} \to \infty$ by virtue of the radiation condition (see Exercise 8.1). Then, after swapping $\mathbf{r}$ and $\mathbf{r}'$, we have

$$\phi_1(\mathbf{r}) = \phi_{inc}(\mathbf{r}) - \int_S dS'\,\hat{n}'\cdot[g_1(\mathbf{r},\mathbf{r}')\nabla'\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}')\nabla' g_1(\mathbf{r},\mathbf{r}')], \quad \mathbf{r}\in V_1. \tag{8.1.9}$$

But if $\mathbf{r}' \notin V_1$ in (5), the second term on the right-hand side of (5) would be zero, for $\mathbf{r}'$ would be in V_2 where the integration is not performed. Therefore, we can write (9) as

$$\left.\begin{matrix}\mathbf{r}\in V_1, & \phi_1(\mathbf{r})\\ \mathbf{r}\in V_2, & 0\end{matrix}\right\} = \phi_{inc}(\mathbf{r}) - \int_S dS'\,\hat{n}'\cdot[g_1(\mathbf{r},\mathbf{r}')\nabla'\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}')\nabla' g_1(\mathbf{r},\mathbf{r}')]. \tag{8.1.10}$$

The above equation is evocative of Huygens' principle. It says that when the observation point $\mathbf{r}$ is in V_1, then the total field $\phi_1(\mathbf{r})$ consists of the incident field, $\phi_{inc}(\mathbf{r})$, and the contribution of field due to surface sources on S. But if the observation point is in V_2, then the surface sources on S generate a field that exactly cancels the incident field $\phi_{inc}(\mathbf{r})$, making the total field in region 2 zero. This fact is the core of the *extinction theorem* (see Born and Wolf 1980).

In (10), $\hat{n}\cdot\nabla\phi_1(\mathbf{r})$ and $\phi_1(\mathbf{r})$ act as surface sources. Moreover, they are impressed on S, creating a field in region 2 that cancels exactly the incident field in region 2 (see Figure 8.1.2).

Applying the same derivation to region 2, we have (see Exercise 8.2)

$$\left.\begin{matrix}\mathbf{r}\in V_2, & \phi_2(\mathbf{r})\\ \mathbf{r}\in V_1, & 0\end{matrix}\right\} = \int_S dS'\,\hat{n}'\cdot[g_2(\mathbf{r},\mathbf{r}')\nabla'\phi_2(\mathbf{r}') - \phi_2(\mathbf{r}')\nabla' g_2(\mathbf{r},\mathbf{r}')]. \tag{8.1.11}$$

The above states that the field in region 2 is due to some surface sources impressed on S. These surface sources generate $\phi_2(\mathbf{r})$ in V_2, but zero field in V_1. This is again evocative of Huygens' principle. [Note that $g_2(\mathbf{r},\mathbf{r}')$ need not have the same form as $g_1(\mathbf{r},\mathbf{r}')$ in (8) (Exercise 8.2).]

[3] Note that this is not the only form satisfying the radiation condition at infinity and satisfying (3) in V_1 (see Exercise 8.1).

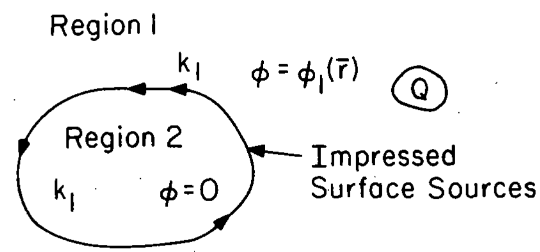


Figure 8.1.2 The illustration of the extinction theorem.

Applying the extinction theorem, integral equations can now be derived. So, using the lower parts of Equations (10) and (11), we have

$$\phi_{inc}(\mathbf{r}) = \int_S dS'\,\hat{n}'\cdot[g_1(\mathbf{r},\mathbf{r}')\nabla'\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}')\nabla' g_1(\mathbf{r},\mathbf{r}')], \quad \mathbf{r}\in V_2, \tag{8.1.12a}$$

$$0 = \int_S dS'\,\hat{n}'\cdot[g_2(\mathbf{r},\mathbf{r}')\nabla'\phi_2(\mathbf{r}') - \phi_2(\mathbf{r}')\nabla' g_2(\mathbf{r},\mathbf{r}')], \quad \mathbf{r}\in V_1. \tag{8.1.12b}$$

Even though $g_1(\mathbf{r},\mathbf{r}')$ and $g_2(\mathbf{r},\mathbf{r}')$ need not be homogeneous-medium Green's functions (Exercise 8.2), homogeneous-medium Green's functions, for simplicity, are usually chosen. Then, the two integral equations above will have four independent unknowns, ϕ_1, ϕ_2, $\hat{n}\cdot\nabla\phi_1$, and $\hat{n}\cdot\nabla\phi_2$ on S. Next, boundary conditions can be used to eliminate two of these four unknowns. Exemplary boundary conditions are

$$\phi_1(\mathbf{r}) = \phi_2(\mathbf{r}), \quad \mathbf{r}\in S, \tag{8.1.13a}$$

$$p_1\hat{n}\cdot\nabla\phi_1(\mathbf{r}) = p_2\hat{n}\cdot\nabla\phi_2(\mathbf{r}), \quad \mathbf{r}\in S. \tag{8.1.13b}$$

Consequently, the integral equations in (12) can be treated as linear operator equations and solved with standard techniques (see Chapter 5). Here, the Green's function $g(\mathbf{r},\mathbf{r}')$ and $\hat{n}\cdot\nabla' g(\mathbf{r},\mathbf{r}')$ are the kernel of the integral equation.

§§*8.1.2 Vector Wave Equation*

Consider the vector electromagnetic wave equation for a two-region problem as shown in Figure 8.1.3. In region 1, the field satisfies the equation

$$\nabla\times\nabla\times\mathbf{E}_1(\mathbf{r}) - \omega^2\mu_1\epsilon_1\mathbf{E}_1(\mathbf{r}) = i\omega\mu_1\mathbf{J}(\mathbf{r}). \tag{8.1.14}$$

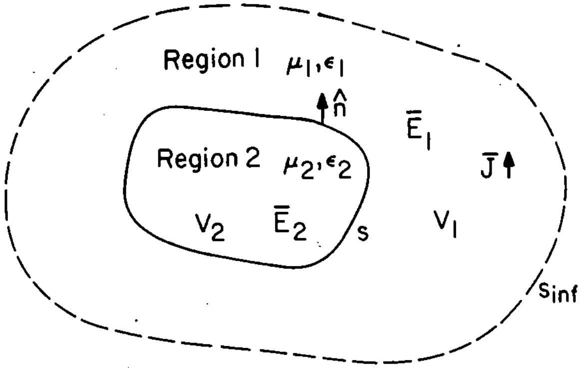


Figure 8.1.3 A two-region problem where a surface integral equation can be derived.

In region 2, the field satisfies

$$\nabla \times \nabla \times \mathbf{E}_2(\mathbf{r}) - \omega^2 \mu_2 \epsilon_2 \mathbf{E}_2(\mathbf{r}) = 0. \tag{8.1.15}$$

Hence, the dyadic Green's functions for region 1 and region 2 respectively are defined by:

$$\nabla \times \nabla \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') - \omega^2 \mu_1 \epsilon_1 \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') = \overline{\mathbf{I}}\,\delta(\mathbf{r} - \mathbf{r}'), \tag{8.1.16}$$

$$\nabla \times \nabla \times \overline{\mathbf{G}}_2(\mathbf{r}, \mathbf{r}') - \omega^2 \mu_2 \epsilon_2 \overline{\mathbf{G}}_2(\mathbf{r}, \mathbf{r}') = \overline{\mathbf{I}}\,\delta(\mathbf{r} - \mathbf{r}'). \tag{8.1.17}$$

After post-multiplying Equation (14) with $\overline{\mathbf{G}}_1$ and pre-multiplying Equation (16) with $\mathbf{E}_1(\mathbf{r})$, subtracting the two equations, and integrating the result over V_1, we have

$$\int_{V_1} dV \left[\nabla \times \nabla \times \mathbf{E}_1(\mathbf{r}) \cdot \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') - \mathbf{E}_1(\mathbf{r}) \cdot \nabla \times \nabla \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}')\right]$$

$$= i\omega\mu_1 \int_{V_1} dV\, \mathbf{J}(\mathbf{r}) \cdot \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') - \mathbf{E}_1(\mathbf{r}'), \qquad \mathbf{r}' \in V_1. \tag{8.1.18}$$

The left-hand side of the above becomes a surface integral using the fact that[4]

$$\nabla \cdot \left\{[\nabla \times \mathbf{E}_1(\mathbf{r})] \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') + \mathbf{E}_1(\mathbf{r}) \times \left[\nabla \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}')\right]\right\}$$
$$= \nabla \times \nabla \times \mathbf{E}_1(\mathbf{r}) \cdot \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') - \mathbf{E}_1(\mathbf{r}) \cdot \nabla \times \nabla \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}'). \tag{8.1.19}$$

[4] We can post-multiply (19) by an arbitrary constant vector **b** to aid in the derivation and cancel the constant vector later. The equality of the volume integral on the left-hand side of (18) to a surface integral is also known as the vector Green's theorem.

Moreover, the integral on the right-hand side of (18) corresponds to the incident wave. Hence, (18) becomes

$$\mathbf{E}_1(\mathbf{r}') = \mathbf{E}_{inc}(\mathbf{r}') + \int_{S+S_{inf}} dS\, \hat{n} \cdot \left\{[\nabla \times \mathbf{E}_1(\mathbf{r})] \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}')\right.$$
$$\left. + \mathbf{E}_1(\mathbf{r}) \times \nabla \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}')\right\}, \quad \mathbf{r}' \in V_1, \tag{8.1.20}$$

where

$$\mathbf{E}_{inc}(\mathbf{r}') = i\omega\mu_1 \int_{V_1} dV\, \mathbf{J}(\mathbf{r}) \cdot \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') = i\omega\mu_1 \int_{V_1} dV\, \overline{\mathbf{G}}_1(\mathbf{r}', \mathbf{r}) \cdot \mathbf{J}(\mathbf{r}) \tag{8.1.21}$$

is the incident field generated by the current $\mathbf{J}(\mathbf{r})$. In the above, we have made use of the reciprocity condition on the dyadic Green's function that [see (1.3.52b) of Chapter 1]

$$\overline{\mathbf{G}}_1^t(\mathbf{r}, \mathbf{r}') = \overline{\mathbf{G}}_1(\mathbf{r}', \mathbf{r}). \tag{8.1.22}$$

Up to this point, $\overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}')$ need only satisfy (16). However, if $\overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}')$ is assumed to be the homogeneous-medium dyadic Green's function given in Chapter 1 and Chapter 7, then $\mathbf{E}_{inc}$ corresponds to the incident field generated by $\mathbf{J}(\mathbf{r})$ in the absence of the scatterer.

Using (22), we deduce that

$$\hat{n} \cdot [\nabla \times \mathbf{E}_1(\mathbf{r})] \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') = \hat{n} \times [\nabla \times \mathbf{E}_1(\mathbf{r})] \cdot \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}')$$
$$= i\omega\mu_1 \overline{\mathbf{G}}_1(\mathbf{r}', \mathbf{r}) \cdot \hat{n} \times \mathbf{H}_1(\mathbf{r}). \tag{8.1.23}$$

Moreover, by assuming that $\overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}')$ is the unbounded homogeneous-medium dyadic Green's function, then[5]

$$\hat{n} \cdot \mathbf{E}_1(\mathbf{r}) \times \nabla \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}') = \hat{n} \times \mathbf{E}_1(\mathbf{r}) \cdot \nabla \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}')$$
$$= -\left[\nabla \times \overline{\mathbf{G}}_1(\mathbf{r}', \mathbf{r})\right] \cdot \hat{n} \times \mathbf{E}_1(\mathbf{r}), \tag{8.1.24}$$

Hence, Equation (20) becomes

$$\mathbf{E}_1(\mathbf{r}') = \mathbf{E}_{inc}(\mathbf{r}') + \int_S dS \left\{i\omega\mu_1 \overline{\mathbf{G}}_1(\mathbf{r}', \mathbf{r}) \cdot \hat{n} \times \mathbf{H}_1(\mathbf{r})\right.$$
$$\left. - \left[\nabla \times \overline{\mathbf{G}}_1(\mathbf{r}', \mathbf{r})\right] \cdot \hat{n} \times \mathbf{E}_1(\mathbf{r})\right\}. \tag{8.1.25}$$

Now, if $\overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}')$ satisfies the radiation condition, the integral over S_{inf} vanishes when $S_{inf} \to \infty$. Note that in (18), if $\mathbf{r}' \notin V_1$, the second term on

[5] The second equality follows from $[\nabla \times \overline{\mathbf{G}}_1(\mathbf{r}, \mathbf{r}')]^t = -\nabla \times \overline{\mathbf{G}}_1(\mathbf{r}', \mathbf{r})$ [Exercise 8.3, and also (1.4.14) of Chapter 1].

the right-hand side of (18) vanishes. Consequently, the analogue of (10) for electromagnetic fields, after swapping $\mathbf{r}$ and $\mathbf{r}'$, is

$$\left.\begin{array}{l}\mathbf{r}\in V_1,\ \mathbf{E}_1(\mathbf{r})\\ \mathbf{r}\in V_2,\ 0\end{array}\right\} = \mathbf{E}_{inc}(\mathbf{r}) + \int_S dS' \left\{ i\omega\mu_1 \overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\hat{n}'\times\mathbf{H}_1(\mathbf{r}') - \left[\nabla'\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\right]\cdot\hat{n}'\times\mathbf{E}_1(\mathbf{r}')\right\}. \quad (8.1.26)$$

Again, the above is similar to Huygens' principle for electromagnetic fields. Furthermore, the lower part of the equation is the vector analogue of the extinction theorem.

In region 2, analogous to Equation (11), similar derivation yields (see Exercise 8.3)

$$\left.\begin{array}{l}\mathbf{r}\in V_2,\ \mathbf{E}_2(\mathbf{r})\\ \mathbf{r}\in V_1,\ 0\end{array}\right\} = -\int_S dS' \left\{ i\omega\mu_2 \overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\cdot\hat{n}'\times\mathbf{H}_2(\mathbf{r}') - \left[\nabla'\times\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\right]\cdot\hat{n}'\times\mathbf{E}_2(\mathbf{r}')\right\}, \quad (8.1.27)$$

where $\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')$ is assumed to be the unbounded homogeneous-medium Green's function.

The lower parts of (26) and (27) form integral equations which are[6]

$$\mathbf{E}_{inc}(\mathbf{r}) = \int_S dS' \left\{ i\omega\mu_1 \overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\hat{n}'\times\mathbf{H}_1(\mathbf{r}') - \left[\nabla'\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\right]\cdot\hat{n}'\times\mathbf{E}_1(\mathbf{r}')\right\}, \quad \mathbf{r}\in V_2, \quad (8.1.28a)$$

$$0 = \int_S dS' \left\{ i\omega\mu_2 \overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\cdot\hat{n}'\times\mathbf{H}_2(\mathbf{r}') - \left[\nabla'\times\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\right]\cdot\hat{n}'\times\mathbf{E}_2(\mathbf{r}')\right\}, \quad \mathbf{r}\in V_1. \quad (8.1.28b)$$

The above, together with the boundary conditions that

$$\hat{n}\times\mathbf{H}_1(\mathbf{r}) = \hat{n}\times\mathbf{H}_2(\mathbf{r}), \quad \hat{n}\times\mathbf{E}_1(\mathbf{r}) = \hat{n}\times\mathbf{E}_2(\mathbf{r}) \quad (8.1.29)$$

on S, can be solved for the surface unknowns $\hat{n}\times\mathbf{E}_1$ and $\hat{n}\times\mathbf{H}_1$. Once the surface fields are known, the field everywhere is derived from the upper parts of Equations (26) and (27). Note that the dyadic Green's functions in (28a) and (28b) need not be homogeneous-medium Green's functions given in Chapters 1 and 7, but homogeneous-medium Green's functions are chosen

[6] Variations of these integral equations are also given by Poggio and Miller (1973) and Ström (1975).

for simplicity (Exercise 8.3). If an arbitrary dyadic function satisfies (16) in region 1 but not the radiation condition at infinity, then the manipulation in (24) is not possible. In this case, the resultant integral equation will have the dyadic Green's functions to the right of the surface sources as in (20) (see next subsection).

Equations (28a) and (28b) are also known as the electric field integral equations (EFIE). By duality, the corresponding magnetic field integral equations (MFIE) can be derived.

§§8.1.3 The Anisotropic, Inhomogeneous Medium Case

Surface integral equations can be derived even when region 1 and region 2 in Figure 8.1.3 consist of anisotropic, inhomogeneous media. In this case, the electric field satisfies the vector wave equations (Section 1.3, Chapter 1)

$$\nabla\times\overline{\boldsymbol{\mu}}_1^{-1}\cdot\nabla\times\mathbf{E}_1(\mathbf{r}) - \omega^2\overline{\boldsymbol{\epsilon}}_1\cdot\mathbf{E}_1(\mathbf{r}) = i\omega\mathbf{J}(\mathbf{r}), \quad \mathbf{r}\in V_1, \quad (8.1.30)$$

$$\nabla\times\overline{\boldsymbol{\mu}}_2^{-1}\cdot\nabla\times\mathbf{E}_2(\mathbf{r}) - \omega^2\overline{\boldsymbol{\epsilon}}_2\cdot\mathbf{E}_2(\mathbf{r}) = 0, \quad \mathbf{r}\in V_2. \quad (8.1.31)$$

Moreover, we can define dyadic Green's functions for regions 1 and 2 respectively as

$$\nabla\times(\overline{\boldsymbol{\mu}}_1^t)^{-1}\cdot\nabla\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}') - \omega^2\overline{\boldsymbol{\epsilon}}_1^t\cdot\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}') = (\overline{\boldsymbol{\mu}}_1^t)^{-1}\overline{\mathbf{I}}\,\delta(\mathbf{r}-\mathbf{r}'), \quad (8.1.32)$$

$$\nabla\times(\overline{\boldsymbol{\mu}}_2^t)^{-1}\cdot\nabla\times\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}') - \omega^2\overline{\boldsymbol{\epsilon}}_2^t\cdot\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}') = (\overline{\boldsymbol{\mu}}_2^t)^{-1}\overline{\mathbf{I}}\,\delta(\mathbf{r}-\mathbf{r}'). \quad (8.1.33)$$

On post-multiplying (30) by $\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}$, we have[7]

$$\left[\nabla\times\overline{\boldsymbol{\mu}}_1^{-1}\cdot\nabla\times\mathbf{E}_1(\mathbf{r})\right]\cdot\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b} - \omega^2\mathbf{E}_1(\mathbf{r})\cdot\overline{\boldsymbol{\epsilon}}_1^t\cdot\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b} = i\omega\mathbf{J}(\mathbf{r})\cdot\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}, \quad (8.1.34)$$

where $\mathbf{b}$ is an arbitrary constant vector. Then, after pre-multiplying (32) by $\mathbf{E}_1(\mathbf{r})$, and post-multiplying it by $\mathbf{b}$, we have

$$\mathbf{E}_1(\mathbf{r})\cdot\nabla\times(\overline{\boldsymbol{\mu}}_1^t)^{-1}\cdot\nabla\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b} - \omega^2\mathbf{E}_1(\mathbf{r})\cdot\overline{\boldsymbol{\epsilon}}_1^t\cdot\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b} = \mathbf{E}_1(\mathbf{r})\,\delta(\mathbf{r}-\mathbf{r}')\cdot(\overline{\boldsymbol{\mu}}_1^t)^{-1}\cdot\mathbf{b}. \quad (8.1.35)$$

Next, integrating the difference of (34) and (35) over V_1, for $\mathbf{r}'\in V_1$ yields

$$\int_{V_1} dV \left[\nabla\times\overline{\boldsymbol{\mu}}_1^{-1}\cdot\nabla\times\mathbf{E}_1(\mathbf{r})\cdot\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b} - \mathbf{E}_1(\mathbf{r})\cdot\nabla\times(\overline{\boldsymbol{\mu}}_1^t)^{-1}\cdot\nabla\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}\right] = i\omega\int_{V_1} dV\,\mathbf{J}(\mathbf{r})\cdot\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b} - \mathbf{E}_1(\mathbf{r}')\cdot(\overline{\boldsymbol{\mu}}_1^t)^{-1}\cdot\mathbf{b}. \quad (8.1.36)$$

[7] Note that the dot product between two vectors $\mathbf{A}\cdot\mathbf{B}$ is actually $\mathbf{A}^t\cdot\mathbf{B}$. Hence, the dot product between $\overline{\boldsymbol{\epsilon}}\cdot\mathbf{E}$ and $\mathbf{B}$ is $\mathbf{E}\cdot\overline{\boldsymbol{\epsilon}}^t\cdot\mathbf{B}$. The transpose sign t over the vector $\mathbf{E}$ is usually ignored.

With the identity that (Exercise 8.4)

$$\nabla\cdot\left\{\left[\overline{\mu}_1^{-1}\cdot\nabla\times\mathbf{E}_1(\mathbf{r})\right]\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}+\mathbf{E}_1(\mathbf{r})\times\left[(\overline{\mu}_1^t)^{-1}\cdot\nabla\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}\right]\right\}$$
$$=\nabla\times\left[\overline{\mu}_1^{-1}\cdot\nabla\times\mathbf{E}_1(\mathbf{r})\right]\cdot\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}-\mathbf{E}_1(\mathbf{r})\cdot\nabla\times\left[(\overline{\mu}_1^t)^{-1}\cdot\nabla\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\mathbf{b}\right],\tag{8.1.37}$$

Equation (36) becomes

$$\mathbf{E}_1(\mathbf{r}')=i\omega\int_{V_1}dV\mathbf{J}(\mathbf{r})\cdot\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\overline{\mu}_1^t(\mathbf{r}')+\int_S dS\,\hat{n}\cdot\left[\overline{\mu}_1^{-1}\cdot\nabla\times\mathbf{E}_1(\mathbf{r})\right.$$
$$\left.\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\overline{\mu}_1^t(\mathbf{r}')+\mathbf{E}_1(\mathbf{r})\times(\overline{\mu}_1^t)^{-1}\cdot\nabla\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\overline{\mu}_1^t(\mathbf{r}')\right],\quad\mathbf{r}'\in V_1.\tag{8.1.38}$$

But if $\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')$ satisfies the radiation condition at infinity, then the surface integral over S_{inf} vanishes by virtue of the radiation condition. This is if the anisotropic, inhomogeneous medium occupies only a finite region in space. Observe now, we have removed the constant vector $\mathbf{b}$ which, up to this point, has been used as a thinking aid. Then, using Maxwell's equations and the appropriate vector identity, Equation (38) becomes

$$\mathbf{E}_1(\mathbf{r}')=i\omega\int_{V_1}dV\mathbf{J}(\mathbf{r})\cdot\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\overline{\mu}_1^t(\mathbf{r}')+\int_S dS\left\{i\omega\hat{n}\times\mathbf{H}_1(\mathbf{r})\cdot\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\overline{\mu}_1^t(\mathbf{r}')\right.$$
$$\left.+\hat{n}\times\mathbf{E}_1(\mathbf{r})\cdot[\overline{\mu}_1^t(\mathbf{r})]^{-1}\cdot\nabla\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\overline{\mu}_1^t(\mathbf{r}')\right\},\quad\mathbf{r}'\in V_1.\tag{8.1.39}$$

Now, the above is just the generalized Huygens' principle for a general anisotropic inhomogeneous medium.

For $\mathbf{r}'\in V_2$, the second term on the right-hand side of (36) vanishes, and (39) is modified accordingly. Consequently, the analogue of Equation (10) is

$$\left.\begin{array}{l}\mathbf{r}\in V_1,\ \mathbf{E}_1(\mathbf{r})\\ \mathbf{r}\in V_2,\ 0\end{array}\right\}=\mathbf{E}_{inc}(\mathbf{r})+\int_S dS'\left\{i\omega\hat{n}'\times\mathbf{H}_1(\mathbf{r}')\cdot\overline{\mathbf{G}}_1(\mathbf{r}',\mathbf{r})\cdot\overline{\mu}_1^t(\mathbf{r})\right.$$
$$\left.+\hat{n}'\times\mathbf{E}_1(\mathbf{r}')\cdot[\overline{\mu}_1^t(\mathbf{r}')]^{-1}\cdot\nabla'\times\overline{\mathbf{G}}_1(\mathbf{r}',\mathbf{r})\cdot\overline{\mu}_1^t(\mathbf{r})\right\},\tag{8.1.40}$$

where $\mathbf{E}_{inc}(\mathbf{r})$ is the first integral on the right-hand side of (39) which corresponds to the field generated by the current source $\mathbf{J}(\mathbf{r})$ in the inhomogeneous medium. Furthermore, the lower part of Equation (40) is just the generalized extinction theorem for anisotropic inhomogeneous media.

In region 2, analogous to Equation (11), a similar derivation yields (see Exercise 8.4)

$$\left.\begin{array}{l}\mathbf{r}\in V_2,\ \mathbf{E}_2(\mathbf{r})\\ \mathbf{r}\in V_1,\ 0\end{array}\right\}=-\int_S dS'\left\{i\omega\hat{n}'\times\mathbf{H}_2(\mathbf{r}')\cdot\overline{\mathbf{G}}_2(\mathbf{r}',\mathbf{r})\cdot\overline{\mu}_2^t(\mathbf{r})\right.$$
$$\left.+\hat{n}'\times\mathbf{E}_2(\mathbf{r}')\cdot[\overline{\mu}_2^t(\mathbf{r}')]^{-1}\cdot\nabla'\times\overline{\mathbf{G}}_2(\mathbf{r}',\mathbf{r})\cdot\overline{\mu}_2^t(\mathbf{r})\right\}.\tag{8.1.41}$$

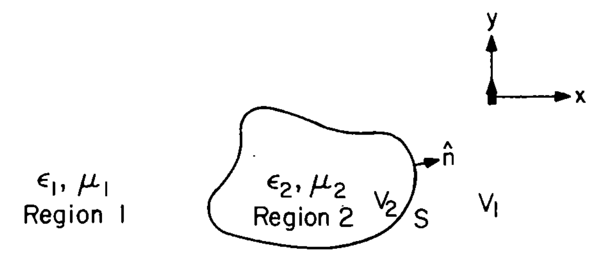


Figure 8.1.4 Inhomogeneity with two piecewise-constant regions for the two-dimensional problem.

Finally, the integral equations for an anisotropic, inhomogeneous medium are

$$-\mathbf{E}_{inc}(\mathbf{r})=\int_S dS'\left\{i\omega\hat{n}'\times\mathbf{H}_1(\mathbf{r}')\cdot\overline{\mathbf{G}}_1(\mathbf{r}',\mathbf{r})\cdot\overline{\mu}_1^t(\mathbf{r})\right.$$
$$\left.+\hat{n}'\times\mathbf{E}_1(\mathbf{r}')\cdot[\overline{\mu}_1^t(\mathbf{r}')]^{-1}\cdot\nabla'\times\overline{\mathbf{G}}_1(\mathbf{r}',\mathbf{r})\cdot\overline{\mu}_1^t(\mathbf{r})\right\},\quad\mathbf{r}\in V_2,\tag{8.1.42a}$$

$$0=\int_S dS'\left\{i\omega\hat{n}'\times\mathbf{H}_2(\mathbf{r}')\cdot\overline{\mathbf{G}}_2(\mathbf{r}',\mathbf{r})\cdot\overline{\mu}_2^t(\mathbf{r})\right.$$
$$\left.+\hat{n}'\times\mathbf{E}_2(\mathbf{r}')\cdot[\overline{\mu}_2^t(\mathbf{r}')]^{-1}\cdot\nabla'\times\overline{\mathbf{G}}_2(\mathbf{r}',\mathbf{r})\cdot\overline{\mu}_2^t(\mathbf{r})\right\},\quad\mathbf{r}\in V_1.\tag{8.1.42b}$$

§§*8.1.4 Two-Dimensional Electromagnetic Case*

If the fields are three-dimensional because of the point nature of the source, but the inhomogeneity is two dimensional and piecewise constant as shown in Figure 8.1.4, we can represent the fields and the source in terms of their Fourier transforms in the z direction. In other words, we can write

$$\mathbf{J}(\mathbf{r})=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk_z\,e^{ik_zz}\mathbf{J}(k_z,\boldsymbol{\rho}),\tag{8.1.43}$$

$$\mathbf{E}(\mathbf{r})=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk_z\,e^{ik_zz}\mathbf{E}(k_z,\boldsymbol{\rho}),\tag{8.1.44}$$

$$\mathbf{H}(\mathbf{r})=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk_z\,e^{ik_zz}\mathbf{H}(k_z,\boldsymbol{\rho}),\tag{8.1.45}$$

where $\boldsymbol{\rho} = \hat{x}x + \hat{y}y$.

Physically, Equations (43) to (45) express the three-dimensional fields as linear superpositions of two-dimensional fields with $e^{ik_z z}$ dependence. But since only two components of the electromagnetic field are truly independent in this case, the field can be represented completely in terms of E_z and H_z. In other words, the field in such a geometry can be decomposed into TM (or E_z) and TE (or H_z) fields. These two-dimensional fields, in the integrands of (43) to (45) in a homogeneous region, satisfy the equations (see Exercise 8.5)

$$[\nabla_s^2 + k^2 - k_z^2]E_z(x,y) = -i\omega\mu J_z + ik_z\frac{\rho}{\epsilon}, \tag{8.1.46a}$$

$$[\nabla_s^2 + k^2 - k_z^2]H_z(x,y) = -(\nabla_s \times \mathbf{J}_s)_z, \tag{8.1.46b}$$

where $\nabla_s = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$. In the above derivations, the $e^{ik_z z}$ dependence of the transformed field has been assumed. Observe that the above equations are applicable in homogeneous region 1 and region 2. Thus, in this manner, a three-dimensional field problem with a two-dimensional inhomogeneity is reduced to a linear superposition of two-dimensional problems with varying k_z. Such a problem is also called a two-and-one-half-dimensional problem.[8] We shall discuss next the derivation of the corresponding surface integral equations.

A horizontal electric dipole pointing in the $\hat{y}$ direction has $\mathbf{J}(\mathbf{r}) = \hat{y}I\ell\delta(x)\,\delta(y)\delta(z)$. Therefore, (43) implies that

$$\begin{aligned}\mathbf{J}(\mathbf{r}) &= \hat{y}I\ell\,\delta(x)\,\delta(y)\,\delta(z)\\ &= \frac{\hat{y}}{2\pi}I\ell\int_{-\infty}^{\infty} dk_z\, e^{ik_z z}\delta(x)\,\delta(y).\end{aligned} \tag{8.1.47}$$

In other words, a point source is a linear superposition of line sources with different $e^{ik_z z}$ dependences. Furthermore, when such a source is embedded in region 1, (46a) and (46b) become

$$(\nabla_s^2 + k_{1s}^2)E_{1z}(\boldsymbol{\rho}) = \frac{I\ell k_z}{\omega\epsilon_1}\,\delta'(y)\,\delta(x), \tag{8.1.48a}$$

$$(\nabla_s^2 + k_{1s}^2)H_{1z}(\boldsymbol{\rho}) = -I\ell\,\delta'(x)\,\delta(y), \tag{8.1.48b}$$

where $k_{1s}^2 = k_1^2 - k_z^2$. The above are also known as the ***reduced wave equations*** because a three-dimensional field problem has been reduced to a two-dimensional problem. Next, we define a two-dimensional Green's function to satisfy the following equation:

$$(\nabla_s^2 + k_{is}^2)G_i(\boldsymbol{\rho},\boldsymbol{\rho}') = -\delta(\boldsymbol{\rho} - \boldsymbol{\rho}'), \tag{8.1.49}$$

[8] The reduction of such a three-dimensional problem to a two-dimensional problem is well known (see, e.g., Wait 1955 in references for Chapter 3; Chuang and Kong 1982).

where $\delta(\boldsymbol{\rho}) = \delta(x)\delta(y)$. On applying concepts similar to the three-dimensional scalar wave equation in arriving at (10), we derive

$$\left.\begin{matrix}\boldsymbol{\rho}\in V_1, & E_{1z}(\boldsymbol{\rho})\\ \boldsymbol{\rho}\in V_2, & 0\end{matrix}\right\} = \frac{I\ell k_z}{\omega\epsilon_1}\frac{\partial}{\partial y'}G_1(\boldsymbol{\rho},\boldsymbol{\rho}'=0) - \int_S dS'\,\hat{n}'\cdot[G_1(\boldsymbol{\rho}',\boldsymbol{\rho})\nabla'_s E_{1z}(\boldsymbol{\rho}') - E_{1z}(\boldsymbol{\rho}')\nabla'_s G_1(\boldsymbol{\rho}',\boldsymbol{\rho})]. \tag{8.1.50}$$

In the above, $G_i(\boldsymbol{\rho},\boldsymbol{\rho}')$ is the two-dimensional unbounded homogeneous-medium Green's function, which is [Equation (2.2.4), Chapter 2]

$$G_i(\boldsymbol{\rho},\boldsymbol{\rho}') = \frac{i}{4}H_0^{(1)}(k_{is}|\boldsymbol{\rho}-\boldsymbol{\rho}'|). \tag{8.1.51}$$

By the same token, a similar equation can be derived for H_{1z}, giving

$$\left.\begin{matrix}\boldsymbol{\rho}\in V_1, & H_{1z}(\boldsymbol{\rho})\\ \boldsymbol{\rho}\in V_2, & 0\end{matrix}\right\} = -I\ell\frac{\partial}{\partial x'}G_1(\boldsymbol{\rho},\boldsymbol{\rho}'=0) - \int_S dS'\,\hat{n}'\cdot[G_1(\boldsymbol{\rho},\boldsymbol{\rho}')\nabla'_s H_{1z}(\boldsymbol{\rho}') - H_{1z}(\boldsymbol{\rho}')\nabla'_s G_1(\boldsymbol{\rho},\boldsymbol{\rho}')]. \tag{8.1.52}$$

Moreover, applying the same concept to region 2 yields (Exercise 8.6)

$$\left.\begin{matrix}\boldsymbol{\rho}\in V_2, & E_{2z}(\boldsymbol{\rho})\\ \boldsymbol{\rho}\in V_1, & 0\end{matrix}\right\} = \int_S dS'\,\hat{n}'\cdot[G_2(\boldsymbol{\rho},\boldsymbol{\rho}')\nabla'_s E_{2z}(\boldsymbol{\rho}') - E_{2z}(\boldsymbol{\rho}')\nabla'_s G_2(\boldsymbol{\rho},\boldsymbol{\rho}')], \tag{8.1.53}$$

$$\left.\begin{matrix}\boldsymbol{\rho}\in V_2, & H_{2z}(\boldsymbol{\rho})\\ \boldsymbol{\rho}\in V_1, & 0\end{matrix}\right\} = \int_S dS'\,\hat{n}'\cdot[G_2(\boldsymbol{\rho},\boldsymbol{\rho}')\nabla'_s H_{2z}(\boldsymbol{\rho}') - H_{2z}(\boldsymbol{\rho}')\nabla'_s G_2(\boldsymbol{\rho},\boldsymbol{\rho}')]. \tag{8.1.54}$$

In the above, $G_2(\boldsymbol{\rho},\boldsymbol{\rho}')$ is the two-dimensional Green's function for region 2. It need only satisfy (49) and need not be of the form (51) (Exercise 8.6).

In the above, E_{iz} and H_{iz} and their normal derivatives are the unknowns. Subsequently, the extinction theorem can be used to write the integral equations as

$$0 = \mathbf{S}_1(\boldsymbol{\rho}) - \int_S dS'\,\hat{n}'\cdot\left\{G_1(\boldsymbol{\rho},\boldsymbol{\rho}')\nabla'_s\begin{bmatrix}E_{1z}(\boldsymbol{\rho}')\\ H_{1z}(\boldsymbol{\rho}')\end{bmatrix} - \begin{bmatrix}E_{1z}(\boldsymbol{\rho}')\\ H_{1z}(\boldsymbol{\rho}')\end{bmatrix}\nabla'_s G_1(\boldsymbol{\rho},\boldsymbol{\rho}')\right\}, \quad \boldsymbol{\rho}\in V_2, \tag{8.1.55}$$

and

$$0 = \int_S dS'\,\hat{n}'\cdot\left\{G_2(\boldsymbol{\rho},\boldsymbol{\rho}')\nabla'_s\begin{bmatrix}E_{2z}(\boldsymbol{\rho}')\\ H_{2z}(\boldsymbol{\rho}')\end{bmatrix} - \begin{bmatrix}E_{2z}(\boldsymbol{\rho}')\\ H_{2z}(\boldsymbol{\rho}')\end{bmatrix}\nabla'_s G_2(\boldsymbol{\rho},\boldsymbol{\rho}')\right\}, \quad \boldsymbol{\rho}\in V_1, \tag{8.1.56}$$

where $\mathbf{S}_1(\boldsymbol{\rho}) = \left[\frac{I\ell k_z}{\omega\epsilon_1}\frac{\partial}{\partial y'}G_1(\boldsymbol{\rho},\boldsymbol{\rho}'=0), -I\ell\frac{\partial}{\partial x'}G_1(\boldsymbol{\rho},\boldsymbol{\rho}'=0)\right]^t$ is the source field in region 1.

Notice that the above consists of four integral equations with eight unknowns, E_{iz}, H_{iz} and their normal derivatives on S. Therefore, the boundary conditions have to be imposed on S to eliminate four of the unknowns. To this end, we require that the tangential fields be continuous. Then,

$$E_{1z} = E_{2z}, \qquad H_{1z} = H_{2z}, \quad \text{on} \quad S. \tag{8.1.57}$$

Furthermore, it is necessary that

$$\hat{n}\times\mathbf{E}_{1s} = \hat{n}\times\mathbf{E}_{2s}, \qquad \hat{n}\times\mathbf{H}_{1s} = \hat{n}\times\mathbf{H}_{2s}, \quad \text{on} \quad S. \tag{8.1.58}$$

Hence, from the equation [see Equation (2.3.17), Chapter 2]

$$\mathbf{E}_s = \frac{i}{k_s^2}[k_z\nabla_s E_z + \omega\mu\nabla_s\times\mathbf{H}_z], \tag{8.1.59}$$

and $\hat{n}\times(\mathbf{E}_{1s}-\mathbf{E}_{2s}) = 0$, we have (see Exercise 8.7)

$$\hat{n}\cdot\nabla_s H_{2z} = -\frac{k_z}{\omega\mu_2}\left(\frac{k_{2s}^2}{k_{1s}^2}-1\right)(\hat{z}\cdot\hat{n}\times\nabla_s)E_{1z} + \frac{\mu_1 k_{2s}^2}{\mu_2 k_{1s}^2}\hat{n}\cdot\nabla_s H_{1z}. \tag{8.1.60}$$

Furthermore, from the duality principle,

$$\hat{n}\cdot\nabla_s E_{2z} = \frac{k_z}{\omega\epsilon_2}\left(\frac{k_{2s}^2}{k_{1s}^2}-1\right)(\hat{z}\cdot\hat{n}\times\nabla_s)H_{1z} + \frac{\epsilon_1 k_{2s}^2}{\epsilon_2 k_{1s}^2}\hat{n}\cdot\nabla_s E_{1z}. \tag{8.1.61}$$

Equations (60) and (61) can be combined as

$$\begin{bmatrix}\hat{n}\cdot\nabla_s E_{2z}\\ \hat{n}\cdot\nabla_s H_{2z}\end{bmatrix} = \overline{\mathbf{M}}\cdot\begin{bmatrix}E_{1z}\\ H_{1z}\end{bmatrix} + \overline{\mathbf{N}}\cdot\begin{bmatrix}\hat{n}\cdot\nabla_s E_{1z}\\ \hat{n}\cdot\nabla_s H_{1z}\end{bmatrix}, \tag{8.1.62}$$

where $M_{11} = M_{22} = N_{12} = N_{21} = 0$,

$$M_{12} = \frac{k_z}{\omega\epsilon_2}\left(\frac{k_{2s}^2}{k_{1s}^2}-1\right)\hat{z}\cdot\hat{n}\times\nabla_s, \qquad N_{11} = \frac{\epsilon_1 k_{2s}^2}{\epsilon_2 k_{1s}^2}, \tag{8.1.63a}$$

$$M_{21} = -\frac{k_z}{\omega\mu_2}\left(\frac{k_{2s}^2}{k_{1s}^2}-1\right)\hat{z}\cdot\hat{n}\times\nabla_s, \quad N_{22} = \frac{\mu_1 k_{2s}^2}{\mu_2 k_{1s}^2}. \tag{8.1.63b}$$

By doing so, and making use of (57) and (62) in (56), then

$$\int_S dS'\left\{G_2(\boldsymbol{\rho},\boldsymbol{\rho}')\overline{\mathbf{N}}\cdot\begin{bmatrix}\hat{n}'\cdot\nabla'_s E_{1z}(\boldsymbol{\rho}')\\ \hat{n}'\cdot\nabla'_s H_{1z}(\boldsymbol{\rho}')\end{bmatrix} - [\hat{n}'\cdot\nabla'_s G_2(\boldsymbol{\rho},\boldsymbol{\rho}') - G_2(\boldsymbol{\rho},\boldsymbol{\rho}')\overline{\mathbf{M}}]\cdot\begin{bmatrix}E_{1z}(\boldsymbol{\rho}')\\ H_{1z}(\boldsymbol{\rho}')\end{bmatrix}\right\} = 0, \quad \boldsymbol{\rho}\in V_1. \tag{8.1.64}$$

In addition, from (55),

$$\int_S dS'\left\{G_1(\boldsymbol{\rho},\boldsymbol{\rho}')\begin{bmatrix}\hat{n}'\cdot\nabla'_s E_{1z}(\boldsymbol{\rho}')\\ \hat{n}'\cdot\nabla'_s H_{1z}(\boldsymbol{\rho}')\end{bmatrix} - \hat{n}'\cdot\nabla'_s G_1(\boldsymbol{\rho},\boldsymbol{\rho}')\begin{bmatrix}E_{1z}(\boldsymbol{\rho}')\\ H_{1z}(\boldsymbol{\rho}')\end{bmatrix}\right\} = \mathbf{S}_1(\boldsymbol{\rho}), \quad \boldsymbol{\rho}\in V_2. \tag{8.1.65}$$

Equation (64) and Equation (65) together constitute two vector integral equations with two vector unknowns. In (64), $\overline{\mathbf{M}}$, being nondiagonal, couples the TE and TM fields together. But if $k_z = 0$, the TE and TM fields are decoupled again, and the problem reduces to two scalar problems (also see Chuang and Kong 1982; Wang and Chew 1989).

§8.2 Solutions by the Method of Moments

Given the integral equations and the boundary conditions, we can solve for the unknown surface field. Then, with the surface field known, the field everywhere can be calculated. The solutions of the integral equation, as such, are pertinent to many scattering problems. Unless the surfaces coincide with some curvilinear coordinate system, the integral equations in general do not have closed-form solutions, and more often than not, the unknowns have to be solved for numerically. Therefore, we shall illustrate the use of two methods, the ***method of moments*** (Harrington 1968) (also known as the method of weighted residuals; see Chapter 5 for references), and the ***extended-boundary-condition method*** (Waterman 1969, 1971) (also known as the ***null-field approach***) to solve such integral equations.

§§8.2.1 Scalar Wave Case

The integral equations in (8.1.12a) and (8.1.12b) can be written symbolically as

$$\mathcal{L}_{11}(\mathbf{r},\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_1(\mathbf{r}') + \mathcal{L}_{12}(\mathbf{r},\mathbf{r}')\,\phi_1(\mathbf{r}') = \phi_{inc}(\mathbf{r}), \qquad \mathbf{r}\in V_2, \tag{8.2.1a}$$

$$\mathcal{L}_{21}(\mathbf{r},\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_2(\mathbf{r}') + \mathcal{L}_{22}(\mathbf{r},\mathbf{r}')\,\phi_2(\mathbf{r}') = 0, \qquad \mathbf{r}\in V_1, \tag{8.2.1b}$$

where the integral operators are

$$\mathcal{L}_{11}(\mathbf{r},\mathbf{r}') = \int_S dS'\,g_1(\mathbf{r},\mathbf{r}'), \quad \mathcal{L}_{12} = -\int_S dS'\,\hat{n}'\cdot\nabla' g_1(\mathbf{r},\mathbf{r}'), \tag{8.2.2a}$$

$$\mathcal{L}_{21}(\mathbf{r},\mathbf{r}') = \int_S dS'\,g_2(\mathbf{r},\mathbf{r}'), \quad \mathcal{L}_{22} = -\int_S dS'\,\hat{n}'\cdot\nabla' g_2(\mathbf{r},\mathbf{r}'). \tag{8.2.2b}$$

$\hat{n}\cdot\nabla\phi$ and ϕ are independent unknowns. But the boundary conditions given by (8.1.13), imply that (1a) and (1b) become

$$\mathcal{L}_{11}(\mathbf{r},\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_1(\mathbf{r}') + \mathcal{L}_{12}(\mathbf{r},\mathbf{r}')\,\phi_1(\mathbf{r}') = \phi_{inc}(\mathbf{r}), \qquad \mathbf{r}\in V_2, \tag{8.2.3a}$$

$$\frac{p_1}{p_2}\mathcal{L}_{21}(\mathbf{r},\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_1(\mathbf{r}') + \mathcal{L}_{22}(\mathbf{r},\mathbf{r}')\,\phi_1(\mathbf{r}') = 0, \qquad \mathbf{r}\in V_1. \tag{8.2.3b}$$

To solve (3a) and (3b) with the method of moments (Chapter 5), we let

$$\hat{n}' \cdot \nabla' \phi_1(\mathbf{r}') = \sum_{n=1}^{N} a_n f_{1n}(\mathbf{r}'), \quad \phi_1(\mathbf{r}') = \sum_{n=1}^{N} b_n f_{2n}(\mathbf{r}'), \tag{8.2.4}$$

where $f_{1n}(\mathbf{r}')$ and $f_{2n}(\mathbf{r}')$ are known basis functions. Then, (3a) and (3b) become

$$\sum_{n=1}^{N} a_n \mathcal{L}_{11}(\mathbf{r},\mathbf{r}') f_{1n}(\mathbf{r}') + \sum_{n=1}^{N} b_n \mathcal{L}_{12}(\mathbf{r},\mathbf{r}') f_{2n}(\mathbf{r}') = \phi_{inc}(\mathbf{r}), \quad \mathbf{r} \in V_2, \tag{8.2.5a}$$

$$\sum_{n=1}^{N} a_n \frac{p_1}{p_2} \mathcal{L}_{21}(\mathbf{r},\mathbf{r}') f_{1n}(\mathbf{r}') + \sum_{n=1}^{N} b_n \mathcal{L}_{22}(\mathbf{r},\mathbf{r}') f_{2n}(\mathbf{r}') = 0, \quad \mathbf{r} \in V_1. \tag{8.2.5b}$$

In (4), we assume the unknown $\hat{n} \cdot \nabla \phi_1(\mathbf{r}')$ is approximated well by $f_{1n}(\mathbf{r}')$ and $\phi_1(\mathbf{r}')$ by $f_{2n}(\mathbf{r}')$. Then, after multiplying (5a) by $w_{1m}(\mathbf{r})$ and (5b) by $w_{2m}(\mathbf{r})$, where $m = 1, \ldots, N$, and integrating over $\mathbf{r}$, we obtain

$$\sum_{n=1}^{N} a_n \langle w_{1m}(\mathbf{r}), \mathcal{L}_{11}(\mathbf{r},\mathbf{r}') f_{1n}(\mathbf{r}') \rangle + \sum_{n=1}^{N} b_n \langle w_{1m}(\mathbf{r}), \mathcal{L}_{12}(\mathbf{r},\mathbf{r}') f_{2n}(\mathbf{r}) \rangle$$
$$= \langle w_{1m}(\mathbf{r}), \phi_{inc}(\mathbf{r}) \rangle, \quad m = 1, \ldots, N, \tag{8.2.6a}$$

$$\sum_{n=1}^{N} a_n \left\langle w_{2m}(\mathbf{r}), \frac{p_1}{p_2} \mathcal{L}_{21}(\mathbf{r},\mathbf{r}') f_{1n}(\mathbf{r}') \right\rangle + \sum_{n=1}^{N} b_n \langle w_{2m}(\mathbf{r}), \mathcal{L}_{22}(\mathbf{r},\mathbf{r}') f_{2n}(\mathbf{r}) \rangle$$
$$= 0, \quad m = 1, \ldots, N. \tag{8.2.6b}$$

The above forms $2N$ linear algebraic equations which yield the $2N$ unknowns, a_n's, $n = 1, \ldots, N$ and b_n's, $n = 1, \ldots, N$. In Equation (6a), we can define $w_{1m}(\mathbf{r})$ anywhere in V_2 and $w_{2m}(\mathbf{r})$ anywhere in V_1. Often, the solution is most stable if $\mathbf{r}$ is chosen close to S. But if $\mathbf{r}$ is chosen to tend to S from either V_1 or V_2, then the singularity of the Green's function has to be properly accounted for.

To see this, notice that if $\mathbf{r} \in S$, the integral arising from the operator $\mathcal{L}_{12}$ and $\mathcal{L}_{22}$, i.e.,

$$\int_S dS' \, \phi(\mathbf{r}') \, \hat{n}' \cdot \nabla' g(\mathbf{r},\mathbf{r}'), \quad \mathbf{r} \in S \tag{8.2.7}$$

does not converge, because the kernel

$$\hat{n}' \cdot \nabla' g(\mathbf{r},\mathbf{r}') \sim O(1/|\mathbf{r}-\mathbf{r}'|^2), \quad |\mathbf{r}-\mathbf{r}'| \to 0 \tag{8.2.8}$$

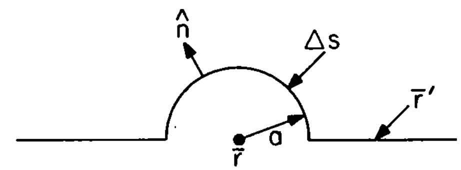


Figure 8.2.1 Diagram for evaluating the residue of a divergent integral.

gives rise to a divergent integral. Such integral equations are also called singular integral equations (see Baker 1977). However, the principal value of the integral and its residue exist, for if $\mathbf{r} \in S$, we can deform the $\mathbf{r}'$ integral around $\mathbf{r}$ and evaluate the integral in the limit when $a \to 0$ as shown in Figure 8.2.1. Therefore,

$$I(\mathbf{r}) = \int_S dS' \, \phi(\mathbf{r}') \, \hat{n}' \cdot \nabla' g(\mathbf{r},\mathbf{r}') = ⨍_S dS' \, \phi(\mathbf{r}') \, \hat{n}' \cdot \nabla' g(\mathbf{r},\mathbf{r}') + \text{Res}, \tag{8.2.9}$$

where

$$⨍_S dS'$$

denotes a principal value integral, while "Res" denotes the residue:

$$\text{Res} = \lim_{a \to 0} \int_{\Delta S} dS' \, \phi(\mathbf{r}') \, \hat{n}' \cdot \nabla' g(\mathbf{r},\mathbf{r}') = \phi(\mathbf{r}) \lim_{a \to 0} \int_{\Delta S} dS' \, \hat{n}' \cdot \nabla' g(\mathbf{r},\mathbf{r}'). \tag{8.2.10}$$

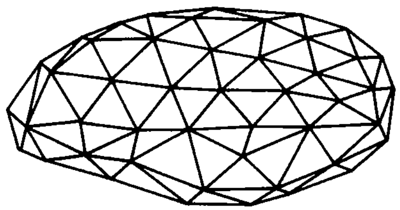
Evaluating the last integral in (10) with $\mathbf{r}$ as the origin yields

$$\lim_{a \to 0} \int_{\Delta S} dS' \, \hat{n}' \cdot \nabla' g(\mathbf{r},\mathbf{r}') = -\int_0^{\pi/2} \int_0^{2\pi} a^2 \sin\theta \, d\theta \, d\phi \frac{1}{4\pi a^2} = -\frac{1}{2}. \tag{8.2.11}$$

Therefore,

$$I(\mathbf{r}) = -\frac{1}{2}\phi(\mathbf{r}) + ⨍_S dS' \, \phi(\mathbf{r}') \, \hat{n}' \cdot \nabla' g(\mathbf{r},\mathbf{r}'). \tag{8.2.12}$$

We can use such principal value integrals to derive integral equations alternative to (8.1.12) (see Exercise 8.8). Note that the residue will have different signs if $\mathbf{r}$ approaches the surface S from different sides. In other words, $I(\mathbf{r})$ is a discontinuous function of $\mathbf{r}$ when $\mathbf{r}$ moves from one side of surface S to the other.



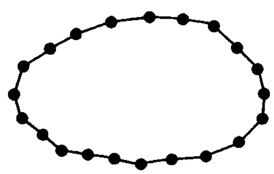


Figure 8.2.2 In the boundary-element method, a smooth surface is approximated by a union of triangles. In two dimensions, a line contour is replaced by a union of line segments.

In (2a) and (2b), if $\mathbf{r}$ and $\mathbf{r}'$ are both defined on the surface S, it can be shown that $\mathcal{L}_{11}$ and $\mathcal{L}_{21}$ are symmetric operators, while $\mathcal{L}_{12}$ and $\mathcal{L}_{22}$ are skew symmetric operators. Therefore, if one chooses $w_{1m} = w_{2m} = f_{1m} = f_{2m}$, as in Galerkin's method (Chapter 5), then the resultant matrix representations of the operators in (6a) and (6b) are either symmetric or skew symmetric (see Exercise 8.9). On the contrary, if w_{1m} and w_{2m} are chosen to be Dirac delta functions, then the solution corresponds to that obtained through the point-matching method or the method of collocation (see Chapter 5 for references).

When the surface S is approximated by a union of triangles or polygons (see Figure 8.2.2), and the expansion functions f_{1n} and f_{2n} are defined over a finite domain (subdomain), e.g., only over the triangles, the method is also known as the ***boundary-element method*** (Brebbia 1978). The boundary-element method is particularly suitable for arbitrarily shaped objects. The subdomain of each element is locally plane, and the choice and construction of basis functions for a subdomain are a lot easier.

§§*8.2.2 The Electromagnetic Case*

Having described a solution technique for the scalar integral equation, we ponder next on the electromagnetic (vector) case. Fortunately, the same idea is easily extended to the vector case for Equations (8.1.28a) and (8.1.28b).

To do this, we let

$$\hat{n}' \times \mathbf{E}_1(\mathbf{r}') = \sum_{n=1}^{N} a_n \mathbf{e}_n(\mathbf{r}'), \quad \hat{n}' \times \mathbf{H}_1(\mathbf{r}') = \sum_{n=1}^{N} b_n \mathbf{h}_n(\mathbf{r}'), \tag{8.2.13}$$

where $\mathbf{e}_n(\mathbf{r}')$ and $\mathbf{h}_n(\mathbf{r}')$ are basis functions that can approximate the vector fields $\hat{n} \times \mathbf{E}_1(\mathbf{r}')$ and $\hat{n} \times \mathbf{H}_1(\mathbf{r}')$ on S fairly well. Testing or weighting functions $\mathbf{w}_{1m}(\mathbf{r})$ and $\mathbf{w}_{2m}(\mathbf{r})$ can be used as in the scalar case.

When the weighting functions are defined over S, as the expansion functions have been, the singularity of the dyadic Green's function must be properly accounted for. Even though the integral

$$\mathbf{I}_1 = \int_S dS'\, \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \cdot \hat{n}' \times \mathbf{H}(\mathbf{r}'), \quad \mathbf{r} \in S \tag{8.2.14}$$

in (8.1.28) seemingly does not converge, it always yields a unique value either via a principal value integral or a vector potential approach (see Exercise 8.10), because $\hat{n} \times \mathbf{H}(\mathbf{r}')$ is an equivalent electric current sheet which produces an electric field that is continuous across this current sheet. However, the integral

$$\mathbf{I}_2 = \int_S dS'\, \nabla' \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \cdot \hat{n}' \times \mathbf{E}(\mathbf{r}'), \quad \mathbf{r} \in S \tag{8.2.15}$$

is undefined, because $\hat{n}' \times \mathbf{E}(\mathbf{r}')$ is an equivalent magnetic current sheet that produces an $\mathbf{E}$ field whose tangential component is discontinuous across the surface S. We shall elaborate this further.

For an unbounded homogeneous medium,

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right) g(\mathbf{r},\mathbf{r}') = \left(\overline{\mathbf{I}} + \frac{\nabla'\nabla'}{k^2}\right) g(\mathbf{r},\mathbf{r}'), \tag{8.2.16a}$$

and

$$\nabla' \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \nabla' \times \overline{\mathbf{I}}\, g(\mathbf{r},\mathbf{r}') = \nabla' g(\mathbf{r},\mathbf{r}') \times \overline{\mathbf{I}}. \tag{8.2.16b}$$

Therefore, Equation (15) becomes

$$\mathbf{I}_2 = \int_S dS'\, [\nabla' g(\mathbf{r},\mathbf{r}')] \times [\hat{n}' \times \mathbf{E}(\mathbf{r}')]. \tag{8.2.17}$$

On multiplying the above by $\hat{n}\times$, and after using the appropriate vector identity, we obtain

$$\begin{aligned} \hat{n} \times \mathbf{I}_2 &= \int_S dS'\, \hat{n} \times \{[\nabla' g(\mathbf{r},\mathbf{r}')] \times [\hat{n}' \times \mathbf{E}(\mathbf{r}')]\} \\ &= -\int_S dS'\, \{\hat{n}' \times \mathbf{E}(\mathbf{r}')\, \hat{n} \cdot \nabla' g(\mathbf{r},\mathbf{r}') - [\hat{n} \cdot \hat{n}' \times \mathbf{E}(\mathbf{r}')]\, \nabla' g(\mathbf{r},\mathbf{r}')\} \end{aligned} \tag{8.2.18}$$

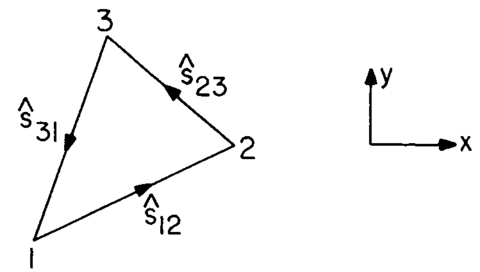


Figure 8.2.3 A triangle with $\hat{s}_{ij}$ defined on the edges.

Then, using the procedure of Equations (9) to (12) (see Exercise 8.10),

$$\hat{n} \times \mathbf{I}_2 = \frac{1}{2}\hat{n} \times \mathbf{E}(\mathbf{r}) - \fint_S dS'\, \hat{n}' \times \mathbf{E}(\mathbf{r}')\, \hat{n} \cdot \nabla' g(\mathbf{r}, \mathbf{r}') + \int_S dS' \left[\hat{n} \cdot \hat{n}' \times \mathbf{E}(\mathbf{r}')\right] \nabla' g(\mathbf{r}, \mathbf{r}') \tag{8.2.19}$$

The first term in (19) is the singularity effect of the integral in (19). The second term in (19) is well defined after the extraction of this singularity. In addition, since the singularity of a dyadic Green's function is a local effect, this technique can also be applied to the integrals in Equations (8.1.42a) and (8.1.42b).

When the boundary-element method is applied to the vector problem, the surface unknowns $\hat{n} \times \mathbf{E}$ and $\hat{n} \times \mathbf{H}$ are expanded over each triangle of a boundary element. But since $\hat{n} \times \mathbf{E}$ and $\hat{n} \times \mathbf{H}$ represent currents, these currents should be continuous across contiguous elements on a surface approximated by a union of triangles (see Figure 8.2.3). For example, we may want the current components normal to the edges of the triangle to be continuous so that no charge accumulates at the edge of the triangle. More specifically, $\hat{n} \times \mathbf{H}$, which is the electric current on a triangle, can be expanded as

$$\mathbf{J}_s = \sum_{i=1}^{3} \mathbf{J}_{si} N_i(x, y), \tag{8.2.20}$$

where xy is the plane of a local coordinate system that contains the triangular patch, and $N_i(x, y)$ is a shape function with value 1 at the i-th node and zero at the other nodes (see Figure 8.2.4). Also, $\mathbf{J}_{si}$ is the value of $\mathbf{J}_s$ at the i-th node, where i ranges from 1 to 3. Hence, $\mathbf{J}_{si}$ can be decomposed into

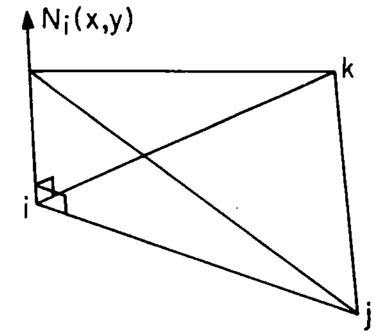


Figure 8.2.4 The shape function $N_i(x, y)$ defined as 1 at one node, zero at the other nodes.

components $\hat{s}_{ij}$ which are unit vectors along the edges of the triangle, as shown in Figure 8.2.3. As a result, the current at the i-th corner of the triangle can be expanded as (see Angkaew et al. 1987)

$$\mathbf{J}_{si} = a_i \hat{s}_{ij} + b_i \hat{s}_{ki}, \tag{8.2.21}$$

where, by simple vector algebra,

$$a_i = \frac{\hat{z} \cdot (\hat{s}_{ki} \times \mathbf{J}_{si})}{\hat{z} \cdot (\hat{s}_{ki} \times \hat{s}_{ij})}, \quad b_i = \frac{\hat{z} \cdot (\hat{s}_{ij} \times \mathbf{J}_{si})}{\hat{z} \cdot (\hat{s}_{ij} \times \hat{s}_{ki})}, \tag{8.2.22}$$

and $\hat{z}$ is normal to the patch surface. With (21) and (22) in mind, (20) is rewritten as

$$\mathbf{J}_s = \sum_{i=1}^{3} [\phi_{i1} \mathbf{N}_{i1}(x, y) + \phi_{i2} \mathbf{N}_{i2}(x, y)], \tag{8.2.23}$$

where

$$\mathbf{N}_{i1} = \frac{\hat{s}_{ij} N_i(x, y)}{\hat{z} \cdot (\hat{s}_{ki} \times \hat{s}_{ij})}, \quad \mathbf{N}_{i2} = \frac{\hat{s}_{ki} N_i(x, y)}{\hat{z} \cdot (\hat{s}_{ij} \cdot x \hat{s}_{ki})}, \tag{8.2.23a}$$

$$\phi_{i1} = \hat{z} \cdot \hat{s}_{ki} \times \mathbf{J}_{si}, \quad \phi_{i2} = \hat{z} \cdot \hat{s}_{ij} \times \mathbf{J}_{si}. \tag{8.2.23b}$$

Note that ϕ_{i1} and ϕ_{i2} are the normal components of $\mathbf{J}_{si}$ (the value of $\mathbf{J}_s$ on the i-th node) on the $\hat{s}_{ki}$ and $\hat{s}_{ij}$ edges respectively. Furthermore, these unknowns are shared by contiguous elements sharing the same edges of the triangles. In this manner, the currents normal to the edges of the triangle are easily rendered continuous from one triangular element to another triangular element, and this method can be used to expand the unknown currents $\hat{n} \times \mathbf{E}$ and $\hat{n} \times \mathbf{H}$ on a triangular patch on S. Subsequently, the matrix representation of the integral operator for the vector electromagnetic field can be obtained (see Exercise 8.11).

For the two-dimensional case, the integral Equations (8.1.64) and (8.1.65) may be written as

$$\overline{\mathcal{L}}_{11}(\boldsymbol{\rho},\boldsymbol{\rho}')\cdot\boldsymbol{\psi}(\boldsymbol{\rho}')+\overline{\mathcal{L}}_{12}(\boldsymbol{\rho},\boldsymbol{\rho}')\cdot\boldsymbol{\phi}(\boldsymbol{\rho}')=\mathbf{S}_1(\boldsymbol{\rho}),\quad \boldsymbol{\rho}'\in S,\quad \boldsymbol{\rho}\in V_2, \tag{8.2.24}$$

$$\overline{\mathcal{L}}_{21}(\boldsymbol{\rho},\boldsymbol{\rho}')\cdot\boldsymbol{\psi}(\boldsymbol{\rho}')+\overline{\mathcal{L}}_{22}(\boldsymbol{\rho},\boldsymbol{\rho}')\cdot\boldsymbol{\phi}(\boldsymbol{\rho}')=0,\quad \boldsymbol{\rho}'\in S,\quad \boldsymbol{\rho}\in V_1. \tag{8.2.25}$$

In order to convert the above into matrix equations, we expand $\boldsymbol{\psi}(\boldsymbol{\rho}')$ and $\boldsymbol{\phi}(\boldsymbol{\rho}')$ in terms of basis functions and find the matrix representations of the integral operators. For example, $\boldsymbol{\psi}(\boldsymbol{\rho}')$ and $\boldsymbol{\phi}(\boldsymbol{\rho}')$ can be expanded as

$$\boldsymbol{\psi}(\boldsymbol{\rho}')=\sum_{n=1}^{N}\overline{\mathbf{g}}_n(\boldsymbol{\rho}')\cdot\mathbf{a}_n,\quad \boldsymbol{\phi}(\boldsymbol{\rho}')=\sum_{n=1}^{N}\overline{\mathbf{g}}_n(\boldsymbol{\rho}')\cdot\mathbf{b}_n, \tag{8.2.26}$$

where

$$\overline{\mathbf{g}}_n(\boldsymbol{\rho}')=\begin{bmatrix} g_{1n}(\boldsymbol{\rho}') & 0\\ 0 & g_{2n}(\boldsymbol{\rho}')\end{bmatrix}. \tag{8.2.26a}$$

Note that in general, we have to expand a vector in terms of a basis set of matrices in order to ensure completeness. For example, if $E_{1z}=\sum_n \alpha_n g_{1n}(\boldsymbol{\rho}')$ and $H_{1z}=\sum_n \beta_n g_{2n}(\boldsymbol{\rho}')$, the proper expansion for $[E_{1z},H_{1z}]^t$ is (also see Exercise 8.12)

$$\begin{bmatrix} E_{1z}\\ H_{1z}\end{bmatrix}=\sum_n\begin{bmatrix} g_{1n}(\boldsymbol{\rho}') & 0\\ 0 & g_{2n}(\boldsymbol{\rho}')\end{bmatrix}\cdot\begin{bmatrix}\alpha_n\\ \beta_n\end{bmatrix}=\sum_n\overline{\mathbf{g}}_n\cdot\boldsymbol{\gamma}_n. \tag{8.2.27}$$

On substituting (27) into (24) and (25) and weighting the equations by $\overline{\mathbf{w}}_m^t(\boldsymbol{\rho})$, where

$$\overline{\mathbf{w}}_m^t(\boldsymbol{\rho})=\begin{bmatrix} w_{1m}(\boldsymbol{\rho}) & 0\\ 0 & w_{2m}(\boldsymbol{\rho})\end{bmatrix},\quad \boldsymbol{\rho}\in S,$$

we obtain

$$\sum_{n=1}^{N}\overline{\mathbf{L}}_{11mn}\cdot\mathbf{a}_n+\sum_{n=1}^{N}\overline{\mathbf{L}}_{12mn}\cdot\mathbf{b}_n=\mathbf{S}_{1m},\quad m=1,\ldots,N, \tag{8.2.28a}$$

$$\sum_{n=1}^{N}\overline{\mathbf{L}}_{21mn}\cdot\mathbf{a}_n+\sum_{n=1}^{N}\overline{\mathbf{L}}_{22mn}\cdot\mathbf{b}_n=0,\quad m=1,\ldots,N, \tag{8.2.28b}$$

where

$$\overline{\mathbf{L}}_{ijmn}=\langle\overline{\mathbf{w}}_m^t(\boldsymbol{\rho}),\overline{\mathbf{L}}_{ij}(\boldsymbol{\rho},\boldsymbol{\rho}'),\overline{\mathbf{g}}_n(\boldsymbol{\rho}')\rangle \tag{8.2.29}$$

is the matrix representation of the operator $\overline{\mathbf{L}}_{ij}$ and

$$\mathbf{S}_{1m}=\langle\overline{\mathbf{w}}_m^t(\boldsymbol{\rho}),\mathbf{S}_1(\boldsymbol{\rho})\rangle. \tag{8.2.30}$$

The system of linear equations in (28) can be solved for $\mathbf{a}_n$ and $\mathbf{b}_n$. Once they are found, the surface fields follow from (26). Then, knowing the surface fields, we can find the field everywhere via the use of Equations (8.1.50) to (8.1.54) in the previous section.

As a final note, firstly, the inner product in (29) usually involves double integrals. Moreover, if $\boldsymbol{\rho}$ and $\boldsymbol{\rho}'$ are both on the surface S, the singularities in $\hat{n}\cdot\nabla_s G_i(\boldsymbol{\rho}-\boldsymbol{\rho}')$ have to be properly accounted for, as has been discussed in Equations (7) to (12). Secondly, if $\overline{\mathbf{w}}_m^t(\boldsymbol{\rho})$ are chosen to comprise Dirac delta functions, then the method corresponds to the point-matching method.

§§8.2.3 *Problem with Internal Resonances*

The surface integral equations discussed previously are easily specialized to impenetrable scatterers. For instance, if an impenetrable scatterer has a Dirichlet boundary condition of $\phi_1(\mathbf{r})=0$ when $\mathbf{r}\in S$, then Equation (8.1.12a) becomes

$$\phi_{inc}(\mathbf{r})=\int_S dS'\,g_1(\mathbf{r},\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_1(\mathbf{r}'),\quad \mathbf{r}\in V_2. \tag{8.2.31}$$

Note that Equation (8.1.12b) is irrelevant now because $\phi_2(\mathbf{r})=0$ inside an impenetrable scatterer. Now, if (31) is imposed on $\mathbf{r}\in S$, severe errors could occur because (31) imposed on S may have a homogeneous solution. In other words,

$$0=\int_S dS'\,g_1(\mathbf{r},\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_1(\mathbf{r}'),\quad \mathbf{r}\in S \tag{8.2.32}$$

could have nontrivial solutions for $\phi_1(\mathbf{r})$, namely, at the internal resonant frequencies of the cavity formed by V_2 (Werner 1963; Schenck 1968; Mitzner 1968; Burton and Miller 1971; Bolomey and Tabbara 1973; Jones 1974; Mittra and Klein 1975; Mautz and Harrington 1978, 1979; Morita 1979; for a review, see Peterson 1990). At these internal resonances, the integral operator defined in (32) has a nonzero nullspace. Hence, its conversion to a matrix form via the method previously described yields an ill-conditioned matrix (see Exercise 8.13).

In addition, the surface source $\hat{n}\cdot\nabla\phi_1(\mathbf{r})$ on S generates no field outside V_2 at resonance [see Exercise 8.13(e)]. Also, from reciprocity (see Chapter 1, Exercise 1.13), the reaction of this surface source with $\phi_{inc}(\mathbf{r})$ is zero since $\phi_{inc}(\mathbf{r})$ is generated by some sources outside V_2. In other words,

$$\int_S dS'\,\phi_{inc}(\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_1(\mathbf{r}')=0 \tag{8.2.33}$$

at the resonant frequencies of V_2 and $\hat{n}\cdot\nabla\phi_1(\mathbf{r})$ is the resonant surface source on S. Because of (33), the incident field is orthogonal to the resonant surface source. Therefore, in principle, this field does not excite the resonant surface source (see Exercise 8.13).

So, at resonances, the resonant surface sources $\hat{n} \cdot \nabla\phi_1(\mathbf{r})$ constitute a nullspace of the integral operator defined in (32). Moreover, the excitation coefficient of this resonant current is zero because it is orthogonal to the incident field as given by (33). What then is the amplitude of the resonant current of the scattering problem at the resonant frequency of the cavity formed by V_2? As it turns out, its amplitude is usually nonzero. In fact, it is similar to the case of finding the value of $\sin(x)/x$ at $x = 0$ (which has a removable singularity at $x = 0$) [see Exercise 8.13(d)]. Consequently, a numerical approximation of (31) is difficult to solve because the "poles" and "zeroes" do not cancel precisely.

A plethora of techniques have been proposed to remove this resonance effect in seeking the solution to (31). For instance, one way is to use the combined-field integral equation approach (see Exercise 8.14; also Mitzner 1968). Yet another way is to avoid imposing (31) only on S, but to impose (31) for all $\mathbf{r} \in V_2$. Then, the equation

$$0 = \int_S dS'\, \hat{n}' \cdot g_1(\mathbf{r},\mathbf{r}')\nabla'\phi_1(\mathbf{r}'), \quad \text{all } \mathbf{r} \in V_2 \tag{8.2.34}$$

will not have a nontrivial solution, since the field generated by the surface source $\hat{n}\cdot\nabla'\phi_1(\mathbf{r}')$ is forced to be zero everywhere in V_2, precluding an internal resonance. This is precisely the spirit of the extended-boundary-condition method to be discuss in the next section. In this manner, Equation (31) becomes

$$\phi_{inc}(\mathbf{r}) = \int_S dS'\, \hat{n}' \cdot g_1(\mathbf{r},\mathbf{r}')\nabla'\phi_1(\mathbf{r}'), \quad \mathbf{r} \in V_2, \tag{8.2.35}$$

which will avoid ill-conditioning. One way of achieving (35), then, is to impose it in a neighborhood interior to S. For example, if the point-matching method is used to solve (35), we will point-match it on S as well as on points just slightly interior to S. This will rid (35) of internal resonances for most practical purposes (see Exercise 8.15).

The scattering solution of a penetrable scatterer governed by (8.1.12) is nontrivial when $\phi_{inc}(\mathbf{r}) = 0$, implying a nonzero nullspace. Here, the physical meaning of the nontrivial solutions corresponds to the resonant modes of the open resonator, formed by two regions with different wave numbers, analogous to a dielectric resonator. Because of radiation damping, however, the resonant frequencies of this open resonator are complex. In other words, the poles of the structure are not on the real ω axis of the complex ω plane. Therefore, for a time harmonic solution where ω is always assumed purely real, the integral operator corresponding to (8.1.12) has no nullspace. Hence,

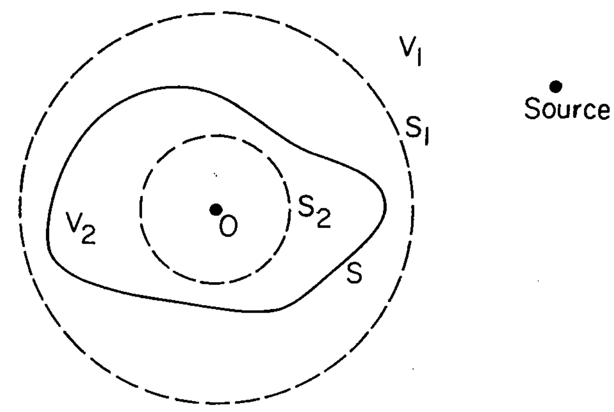


Figure 8.3.1 The surfaces for the extended-boundary-condition method or the null-field approach. V_1 is the volume outside S, while V_2 is the volume inside S.

the problems that plague the solution of (31) do not exist, unless the Q of the resonance mode is so very high that the poles of the structure lie very close to the real ω axis. (Note that the above discussions apply to the vector electromagnetic case too.)

§8.3 Extended-Boundary-Condition Method

The extended-boundary-condition (EBC) method, developed by Waterman (1969, 1971), is also known as the null-field approach. It is an alternative to solve the surface integral equation. In this method, the integral equations are imposed not on the surface S, but on some surfaces S_1 and S_2 away from S as shown in Figure 8.3.1, in order to simplify the solutions. In this section, we shall discuss this method for solving the surface integral equation for the scalar wave case first and discuss the electromagnetics case later.

§§*8.3.1 The Scalar Wave Case*

The scalar integral equations are

$$\phi_{inc}(\mathbf{r}) = \int_S dS' \left[g_1(\mathbf{r},\mathbf{r}')\,\hat{n}' \cdot \nabla'\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}')\,\hat{n}' \cdot \nabla' g_1(\mathbf{r},\mathbf{r}')\right], \quad \mathbf{r} \in V_2, \tag{8.3.1}$$

$$0 = \int_S dS' \left[g_2(\mathbf{r},\mathbf{r}')\,\hat{n}' \cdot \nabla'\phi_2(\mathbf{r}') - \phi_2(\mathbf{r}')\,\hat{n}' \cdot \nabla' g_2(\mathbf{r},\mathbf{r}')\right], \quad \mathbf{r} \in V_1, \tag{8.3.2}$$

where Equation (1) is valid for $\mathbf{r}$ anywhere in V_2 bounded by S and Equation (2) for $\mathbf{r}$ anywhere in V_1 outside S. As such, it is often convenient to impose the integral equation in (1) on S_2, a spherical surface in V_2, and to impose integral equation in (2) on S_1, a spherical surface in V_1. This is further motivated by a subsequent simplification in the solutions of the integral equation. In this method, the integral equations are imposed on surfaces away from the actual surface of the object, and hence, its name, the extended-boundary-condition method.

The integral equations on the spherical surfaces simplify if spherical harmonics are used in the expansion of the field. For instance, via the addition theorem, the Green's function in a homogeneous medium can be expanded as (see Exercise 8.16)[9]

$$g(\mathbf{r},\mathbf{r}') = ik\sum_n \psi_n(k,\mathbf{r}_>)\Re e\psi_n(k,\mathbf{r}_<), \tag{8.3.3}$$

where $\psi_n(k,\mathbf{r})$ represents an outgoing wave spherical harmonic, and $\Re e\psi_n(k,\mathbf{r})$ is the ***regular*** part of $\psi_n(k,\mathbf{r})$.[10] $\mathbf{r}_>$ represents the larger of the $\mathbf{r}$ and $\mathbf{r}'$, and $\mathbf{r}_<$ is the smaller of the $\mathbf{r}$ and $\mathbf{r}'$ in magnitudes. In addition, since the incident wave is regular about the origin, it can be expanded as the regular spherical wave functions, i.e.,

$$\phi_{inc}(\mathbf{r}) = \sum_n a_n\Re e\psi_n(k_1,\mathbf{r}). \tag{8.3.4}$$

As such, Equations (1) and (2) become

$$\sum_n a_n\Re e\psi_n(k_1,\mathbf{r}) = ik_1\sum_n \Re e\psi_n(k_1,\mathbf{r})\int_S dS'\,[\psi_n(k_1,\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}')\,\hat{n}'\cdot\nabla'\psi_n(k_1,\mathbf{r}')], \quad \mathbf{r}\in S_2, \tag{8.3.5}$$

$$0 = ik_2\sum_n \psi_n(k_2,\mathbf{r})\int_S dS'\,[\Re e\psi_n(k_2,\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_2(\mathbf{r}') - \phi_2(\mathbf{r}')\,\hat{n}'\cdot\nabla'\Re e\psi_n(k_2,\mathbf{r}')], \quad \mathbf{r}\in S_1. \tag{8.3.6}$$

In the above, (5) is actually valid for $\mathbf{r}$ anywhere in the volume bounded by S_2, and (6) is actually valid for $\mathbf{r}$ anywhere in the volume outside S_1. Consequently, from the orthogonality of the spherical harmonics, we deduce that

$$a_n = ik_1\int_S dS'\,[\psi_n(k_1,\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}')\,\hat{n}'\cdot\nabla'\psi_n(k_1,\mathbf{r}')],\ \forall n, \tag{8.3.7}$$

$$0 = ik_2\int_S dS'\,[\Re e\psi_n(k_2,\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_2(\mathbf{r}') - \phi_2(\mathbf{r}')\,\hat{n}'\cdot\nabla'\Re e\psi_n(k_2,\mathbf{r}')],\ \forall n. \tag{8.3.8}$$

[9] The factor k in (3) is replaced by 1/4 in two dimensions.

[10] If $\psi_n(k,\mathbf{r})$ consists of a Hankel function, which is singular at $\mathbf{r}=0$, then $\Re e\psi_n(k,\mathbf{r})$ consists of a Bessel function, which is regular at $\mathbf{r}=0$.

Observe that the integral equations are greatly simplified now and do not involve $\mathbf{r}$ at all. To solve this new integral equations, we expand the surface unknowns $\hat{n}\cdot\nabla'\phi(\mathbf{r}')$ and $\phi(\mathbf{r}')$ in terms of a basis set to convert (7) and (8) into matrix equations. A clever way of expanding the surface unknowns is to let

$$\phi_2(\mathbf{r}') = \sum_m \alpha_m\Re e\psi_m(k_2,\mathbf{r}'), \tag{8.3.9}$$

$$\hat{n}'\cdot\nabla'\phi_2(\mathbf{r}') \doteq \sum_m \beta_m\hat{n}'\cdot\nabla'\Re e\psi_m(k_2,\mathbf{r}'). \tag{8.3.10}$$

But this is only rigorously valid if $\Re e\psi_m(k_2,\mathbf{r}')$ and $\hat{n}\cdot\nabla'\Re e\psi_m(k_2,\mathbf{r}')$ are complete on S. As it turns out, they are complete except at the internal resonances of the cavity formed by V_2 with wave number k_2 (see Exercise 8.17; also see Waterman 1969). In this case, it may seem obvious that $\alpha_m=\beta_m$, but this may be inconsistent with (8). To check this consistency, we use (9) and (10) in (8) to yield

$$0 = \sum_m \beta_m\int_S dS'\,[\Re e\psi_n(k_2,\mathbf{r}')\,\hat{n}'\cdot\nabla'\Re e\psi_m(k_2,\mathbf{r}')] - \sum_m \alpha_m\int_S dS'\,[\Re e\psi_m(k_2,\mathbf{r}')\,\hat{n}'\cdot\nabla'\Re e\psi_n(k_2,\mathbf{r}')]. \tag{8.3.11}$$

At this point, it is not obvious at all that α_m should be β_m. However, by integrating

$$\nabla'\cdot[\Re e\psi_n(k_2,\mathbf{r}')\nabla'\Re e\psi_m(k_2,\mathbf{r}') - \Re e\psi_m(k_2,\mathbf{r}')\nabla'\Re e\psi_n(k_2,\mathbf{r}')] \tag{8.3.12}$$

over a volume V bounded by S and S_2, and applying Gauss' theorem and the fact that (12) is zero and that the integral over S_2 is zero due to the orthogonality of spherical harmonics on a spherical surface, we conclude that (see Exercise 8.18)

$$\int_S dS'\,[\Re e\psi_n(k_2,\mathbf{r}')\,\hat{n}\cdot\nabla'\Re e\psi_m(k_2,\mathbf{r}')] = \int_S dS'\,\Re e\psi_m(k_2,\mathbf{r}')\,\hat{n}'\cdot\nabla'\Re e\psi_n(k_2,\mathbf{r}'). \tag{8.3.13}$$

This, when used in (11), indeed implies that $\alpha_m=\beta_m$. Hence, the reward for the clever choice of expansion functions in (9) and (10) is that it solves (8) immediately. As a result, Equation (7), with the use of (9) and (10), and the boundary conditions (8.1.13), becomes

$$a_n = ik_1\sum_m \alpha_m\int_S dS'\left[\psi_n(k_1,\mathbf{r}')\,\hat{n}'\cdot\nabla'\Re e\psi_m(k_2,\mathbf{r}')\frac{p_2}{p_1} - \Re e\psi_m(k_2,\mathbf{r}')\,\hat{n}'\cdot\nabla'\psi_n(k_1,\mathbf{r}')\right]. \tag{8.3.14}$$

Then, Equation (14) is of the form

$$a_n = i\sum_m \alpha_m Q_{nm}, \tag{8.3.15}$$

where

$$Q_{nm} = k_1 \int_S dS' \Big[\psi_n(k_1,\mathbf{r}')\,\hat{n}'\cdot\nabla'\Re e\psi_m(k_2,\mathbf{r}')\frac{p_2}{p_1} - \Re e\psi_m(k_2,\mathbf{r}')\,\hat{n}'\cdot\nabla'\psi_n(k_1,\mathbf{r}')\Big]. \tag{8.3.16}$$

Note that Equation (15) is in theory an infinite-dimensional matrix equation. But in practice, α_m is solvable from (15) in terms of a_n, the amplitudes of the incident field, by truncating the infinite-dimensional matrix equation.

In the extended-boundary-condition method, the fields are extinct only inside the circle S_2 and outside the circle S_1. In actual fact, the surface sources impressed on S have to extinct the pertinent field everywhere inside S or everywhere outside S. As such, the extended-boundary-condition method provides a solution which sometimes is an approximation of the actual solution, albeit such an approximation is very good in a number of cases.

Because the testing surfaces S_1 and S_2 are away from the surfaces in the extended-boundary-condition method, the matrix equation (15) becomes very ill-conditioned if the maximum distances of S_1 and S_2 from S are large. For instance, this is the case for an elongated object or an object with high corrugations. Physically, the ill-conditioning arises because the surface sources on S generate fields which are localized in the vicinity of S for such surfaces (in the case of a planar, corrugated surface, this would be an evanescent type wave). Accordingly, information on S within such fields is greatly diminished on S_1 and S_2. Therefore, imposing the integral equations (1) and (2) on S_1 and S_2 results in a set of ill-conditioned equations. [The convergence of the Waterman EBC method has been studied by Bolomey and Wirgin (1974) and Bates and Wall (1977).] Despite this, the EBC method is attractive for many applications because it gives a simpler set of equations. It is particularly convenient for a scatterer where the fields around it are expandable in spherical harmonics, cylindrical harmonics, or Floquet modes (e.g., periodic rough surfaces).

The EBC method is also easily adaptable to impenetrable objects. In this case, we need to solve only Equation (7) with either the homogeneous Dirichlet or Neumann boundary condition, or the impedance boundary condition. By the same token as in (9) and (10), one can expand $\hat{n}\cdot\nabla\phi(\mathbf{r})$ in terms of $\hat{n}\cdot\nabla\Re e\psi_n(k_2,\mathbf{r})$, and $\phi(\mathbf{r})$ in terms of $\Re e\psi_n(k_2,\mathbf{r})$.[11] But at the internal resonance of the cavity formed by S, the set $\hat{n}\cdot\nabla\Re e\psi_n(k_2,\mathbf{r})$ or $\Re e\psi_n(k_2,\mathbf{r})$ is incomplete on S as mentioned previously (see Exercises 8.17, 8.19). Then, the Q_{nm} matrix thus derived is ill-conditioned for quite a different reason from those discussed in Subsection 8.2.3 (see Exercise 8.20). Hence, this internal resonance can be overcome by using a complete set to expand the surface sources (see Waterman 1969).

[11] k_2 may be chosen to be k_1 in this case.

§§8.3.2 The Electromagnetic Wave Case

The extended-boundary-condition method is easily generalized to the vector electromagnetics case for solving the integral equations from (8.1.28) (Waterman 1971; Barber and Yeh 1975; also see papers in Kerker 1988). The vector surface integral equations are

$$\mathbf{E}_{inc}(\mathbf{r}) = \int_S dS'\,\big[i\omega\mu_1\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\hat{n}'\times\mathbf{H}_1(\mathbf{r}') - \nabla'\times\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')\cdot\hat{n}'\times\mathbf{E}_1(\mathbf{r}')\big],\quad \mathbf{r}\in V_2, \tag{8.3.17}$$

$$0 = \int_S dS'\,\big[i\omega\mu_2\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\cdot\hat{n}'\times\mathbf{H}_2(\mathbf{r}') - \nabla'\times\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')\cdot\hat{n}'\times\mathbf{E}_2(\mathbf{r}')\big],\quad \mathbf{r}\in V_1. \tag{8.3.18}$$

In this case, the unbounded homogeneous-medium dyadic Green's function is expanded as [see Exercise 8.21 and Equation (7.3.40)]

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = ik\sum_n \boldsymbol{\psi}_n(k,\mathbf{r}_>)\Re e\boldsymbol{\psi}_n(k,\mathbf{r}_<),\qquad \mathbf{r}_> \neq \mathbf{r}_<, \tag{8.3.19}$$

where $\boldsymbol{\psi}_n(k,\mathbf{r})$'s are vector wave functions for outgoing waves.

By the same token as (4), the incident wave is expanded as the regular vector wave functions

$$\mathbf{E}_{inc}(\mathbf{r}) = \sum_n a_n\Re e\boldsymbol{\psi}_n(k_1,\mathbf{r}). \tag{8.3.20}$$

Consequently, (17) and (18) become (Exercise 8.22)

$$a_n = ik_1\int_S dS'\,[i\omega\mu_1\boldsymbol{\psi}_n(k_1,\mathbf{r}')\cdot\hat{n}'\times\mathbf{H}_1(\mathbf{r}') - \nabla'\times\boldsymbol{\psi}_n(k_1,\mathbf{r}')\cdot\hat{n}'\times\mathbf{E}_1(\mathbf{r}')], \tag{8.3.21}$$

$$0 = ik_2\int_S dS'\,[i\omega\mu_2\Re e\boldsymbol{\psi}_n(k_2,\mathbf{r}')\cdot\hat{n}'\times\mathbf{H}_2(\mathbf{r}') - \nabla'\times\Re e\boldsymbol{\psi}_n(k_2,\mathbf{r}')\cdot\hat{n}'\times\mathbf{E}_2(\mathbf{r}')]. \tag{8.3.22}$$

Next, we expand

$$\hat{n}' \times \mathbf{E}_2(\mathbf{r}') = \sum_m \alpha_m \hat{n}' \times \Re e\psi_m(k_2, \mathbf{r}'), \qquad \mathbf{r}' \in S, \tag{8.3.23}$$

$$i\omega\mu_2 \hat{n}' \times \mathbf{H}_2(\mathbf{r}') = \sum_m \beta_m \hat{n}' \times \nabla' \times \Re e\psi_m(k_2, \mathbf{r}'), \quad \mathbf{r}' \in S, \tag{8.3.24}$$

because $\hat{n}' \times \Re e\psi_m(k_2, \mathbf{r}')$ and $\hat{n}' \times \nabla' \times \Re e\psi_m(k_2, \mathbf{r}')$ are complete on S (see Exercise 8.23). Then, Equation (22) becomes

$$0 = \sum_m \int_S dS' \, [\beta_m \Re e\psi_n(k_2, \mathbf{r}') \cdot \hat{n}' \times \nabla' \times \Re e\psi_m(k_2, \mathbf{r}') - \alpha_m \nabla' \times \Re e\psi_n(k_2, \mathbf{r}') \cdot \hat{n}' \times \Re e\psi_m(k_2, \mathbf{r}')]. \tag{8.3.25}$$

To prove $\alpha_m = \beta_m$, we take the volume integral of

$$\nabla' \cdot \{[\nabla' \times \Re e\psi_m(k_2, \mathbf{r}')] \times \Re e\psi_n(k_2, \mathbf{r}') + \Re e\psi_m(k_2, \mathbf{r}') \times [\nabla' \times \Re e\psi_n(k_2, \mathbf{r}')]\} \tag{8.3.26}$$

over a volume bounded by S and S_2 and note that the above is zero and that the surface integral over S_2 vanishes for $n \neq m$. Hence, we conclude that (see Exercise 8.24)

$$\int_S dS' \, \hat{n}' \cdot [\nabla' \times \Re e\psi_m(k_2, \mathbf{r}')] \times \Re e\psi_n(k_2, \mathbf{r}') = \int_S dS' \hat{n}' \cdot [\nabla' \times \Re e\psi_n(k_2, \mathbf{r}')] \times \Re e\psi_m(k_2, \mathbf{r}') \tag{8.3.27}$$

or that $\alpha_m = \beta_m$ in (25) and then, in (23) and (24). Then, after making use of the continuity of $\hat{n} \times \mathbf{E}$ and $\hat{n} \times \mathbf{H}$ on S, (21) becomes

$$\begin{aligned} a_n = ik_1 \sum_m \alpha_m \int_S dS' \, & [(\mu_1/\mu_2)\hat{n}' \cdot \nabla' \times \Re e\psi_m(k_2, \mathbf{r}') \times \psi_n(k_1, \mathbf{r}') \\ & - \hat{n}' \cdot \Re e\psi_m(k_2, \mathbf{r}') \times \nabla' \times \psi_n(k_1, \mathbf{r}')] \\ = i \sum_m \alpha_m Q_{nm}, & \end{aligned} \tag{8.3.28}$$

where

$$Q_{nm} = k_1 \int_S dS' \, [(\mu_1/\mu_2)\hat{n}' \cdot \nabla' \times \Re e\psi_m(k_2, \mathbf{r}') \times \psi_n(k_1, \mathbf{r}') - \hat{n}' \cdot \Re e\psi_m(k_2, \mathbf{r}') \times \nabla' \times \psi_n(k_1, \mathbf{r}')]. \tag{8.3.29}$$

From Equation (28), we can solve for α_m's which then yield the surface unknowns. The matrix Equation (28) suffers from ill-conditioning for the same reason given for the scalar wave case when the object is elongated or the surface of the object is convoluted. In addition, for impenetrable objects, (28) suffers from resonance effects as in the scalar wave case.

§8.4 The Transition and Scattering Matrices

Once a scattering problem is solved, the scattered field everywhere can be found. For instance, in the extended-boundary-condition method for the scalar wave, once the α_m's are found in Equation (8.3.15) by truncating the infinite-dimensional matrix equation, the surface fields are found using (8.3.9) and (8.3.10). The scattered field from (8.1.9) then becomes

$$\begin{aligned} \phi_{sca}(\mathbf{r}) = \sum_n f_n \psi_n(k_1, \mathbf{r}) = -\sum_{nm} ik_1 \alpha_m \psi_n(k_1, \mathbf{r}) \int_S dS' \Big[& \Re e\psi_n(k_1, \mathbf{r}') \\ \hat{n}' \cdot \nabla' \Re e\psi_m(k_2, \mathbf{r}') \frac{p_2}{p_1} - \Re e\psi_m(k_2, \mathbf{r}') \, \hat{n}' \cdot \nabla' \Re e \cdot \psi_n(k_1, \mathbf{r}') & \Big]. \end{aligned} \tag{8.4.1}$$

From the above, we readily deduce that

$$f_n = -i \sum_m \alpha_m \Re e Q_{nm}, \tag{8.4.2}$$

where Q_{nm} is defined in (8.3.16), and $\Re e$ implies the "regular part of." In other words, $\Re e Q_{nm}$ will convert all the Hankel functions in Q_{nm} into Bessel functions. Consequently, from (8.3.15) and (2), we have

$$\mathbf{a} = i\overline{\mathbf{Q}} \cdot \boldsymbol{\alpha}, \tag{8.4.3a}$$

$$\mathbf{f} = -i \left(\Re e \overline{\mathbf{Q}}\right) \cdot \boldsymbol{\alpha}, \tag{8.4.3b}$$

where $\mathbf{a}$, $\boldsymbol{\alpha}$, and $\mathbf{f}$ are vectors containing a_n, α_n, and f_n respectively, and $\overline{\mathbf{Q}}$ is a matrix with elements Q_{mn}. On eliminating $\boldsymbol{\alpha}$ in the above, we obtain

$$\mathbf{f} = -\left(\Re e \overline{\mathbf{Q}}\right) \cdot \overline{\mathbf{Q}}^{-1} \cdot \mathbf{a}. \tag{8.4.4}$$

A transition matrix $\overline{\mathbf{T}}$ can be defined to relate the scattered wave amplitude to the incoming wave amplitude such that (Waterman 1969, 1971)

$$\mathbf{f} = \overline{\mathbf{T}} \cdot \mathbf{a}, \tag{8.4.5}$$

where

$$\overline{\mathbf{T}} = -\left(\Re e \overline{\mathbf{Q}}\right) \cdot \overline{\mathbf{Q}}^{-1}. \tag{8.4.6}$$

Hence, the total field then becomes

$$\begin{aligned} \phi &= \sum_n [a_n \Re e\psi_n(k_1, \mathbf{r}) + f_n \psi_n(k_1, \mathbf{r})] \\ &= \sum_n \left[a_n \Re e\psi_n(k_1, \mathbf{r}) + \left(\sum_m T_{nm} a_m \right) \psi_n(k_1, \mathbf{r}) \right]. \end{aligned} \tag{8.4.7}$$

Notice that the preceding equation is the same as

$$\phi(\mathbf{r}) = \left[\Re e\boldsymbol{\psi}^t(k_1,\mathbf{r}) + \boldsymbol{\psi}^t(k_1,\mathbf{r})\cdot\overline{\mathbf{T}}\right]\cdot\mathbf{a}, \tag{8.4.8}$$

where $\boldsymbol{\psi}(k_1,\mathbf{r})$ is a column vector containing $\psi_n(k_1,\mathbf{r})$.

If a scattering matrix $\overline{\mathbf{S}}$ is defined which is related to $\overline{\mathbf{T}}$ as

$$\overline{\mathbf{S}} = \overline{\mathbf{I}} + 2\overline{\mathbf{T}}, \tag{8.4.9}$$

then (8) becomes

$$\phi(\mathbf{r}) = \left[\Re e\boldsymbol{\psi}^t(k_1,\mathbf{r}) - \frac{1}{2}\boldsymbol{\psi}^t(k_1,\mathbf{r}) + \frac{1}{2}\boldsymbol{\psi}^t(k_1,\mathbf{r})\cdot\overline{\mathbf{S}}\right]\cdot\mathbf{a}. \tag{8.4.10}$$

Consider the fact that

$$\Re e\psi_n(k_1,\mathbf{r}) = \frac{1}{2}\psi_n(k_1,\mathbf{r}) + \frac{1}{2}\psi_n(-k_1,\mathbf{r}), \tag{8.4.11}$$

i.e., a standing wave $\Re e\psi_n(k_1,\mathbf{r})$ can be written as a linear superposition of an outgoing wave $\frac{1}{2}\psi_n(k_1,\mathbf{r})$ plus an incoming wave $\frac{1}{2}\psi_n(-k_1,\mathbf{r})$. Then, (10) becomes

$$\phi(\mathbf{r}) = \frac{1}{2}\left[\boldsymbol{\psi}^t(-k_1,\mathbf{r}) + \boldsymbol{\psi}^t(k_1,\mathbf{r})\cdot\overline{\mathbf{S}}\right]\cdot\mathbf{a}. \tag{8.4.12}$$

Therefore, the scattering matrix $\overline{\mathbf{S}}$ relates the amplitude of the scattered wave to the incoming wave.

Using reciprocity, it can be proven that $\overline{\mathbf{T}}$ is a symmetric matrix (see Exercise 8.25). Thus,

$$\overline{\mathbf{T}}^t = \overline{\mathbf{T}}, \quad \overline{\mathbf{S}}^t = \overline{\mathbf{S}}. \tag{8.4.13}$$

Moreover, energy conservation implies that (see Exercise 8.26)

$$\overline{\mathbf{S}}^\dagger\cdot\overline{\mathbf{S}} = \overline{\mathbf{I}}, \quad \text{or} \quad \overline{\mathbf{S}}^*\cdot\overline{\mathbf{S}} = \overline{\mathbf{I}}, \tag{8.4.14a}$$

and

$$\overline{\mathbf{T}}^\dagger\cdot\overline{\mathbf{T}} = -\Re e\overline{\mathbf{T}}. \tag{8.4.14b}$$

The above are useful checks for the correctness of the $\overline{\mathbf{T}}$ and $\overline{\mathbf{S}}$ matrices when they are computed. Finally, even though the $\overline{\mathbf{T}}$ and $\overline{\mathbf{S}}$ matrices have been derived here using the EBC solution as an illustration, they can in theory be defined once the scattering solution is known, regardless of the method of solution.

§8.5 The Method of Rayleigh's Hypothesis

A method very closely related to the EBC method is the method of Rayleigh's hypothesis (Rayleigh 1894, 1897, 1907). Even though this method does not involve integral equations, it is worthy of discussion because of its close relationship to the EBC method.

Consider the geometry shown in Figure 8.3.1; the field outside the surface S_1 can be expanded in terms of the incident and scattered waves, which are

$$\phi_{inc}(\mathbf{r}) = \sum_n a_n\Re e\psi_n(k_1,\mathbf{r}), \tag{8.5.1a}$$

$$\phi_{sca}(\mathbf{r}) = \sum_n f_n\psi_n(k_1,\mathbf{r}). \tag{8.5.1b}$$

Note that we have expanded the incident field in terms of standing waves but the scattered field in terms of outgoing waves. Next, the field inside the scatterer is expanded again in terms of a standing wave of the form

$$\phi_2(\mathbf{r}) = \sum_n \alpha_n\Re e\psi_n(k_2,\mathbf{r}) \tag{8.5.2}$$

for $\mathbf{r}$ inside S_2. In addition to this, Rayleigh's hypothesis assumes that (1b) and (2) are valid on S as well. But this is not at all clear because, in the region bounded by S_1 and S, it is not obvious if all the waves are outgoing as expressed by (1b). Moreover, in the region bounded by S_2 and S, it is not obvious if all the waves are standing waves.

In spite of this, we assume the validity of Rayleigh's hypothesis and match boundary conditions on S. Consequently, the continuity of the potential implies

$$\sum_n[a_n\Re e\psi_n(k_1,\mathbf{r}) + f_n\psi_n(k_1,\mathbf{r})] = \sum_n \alpha_n\Re e\psi_n(k_2,\mathbf{r}), \quad \mathbf{r}\in S. \tag{8.5.3}$$

Furthermore, the boundary condition on the normal derivatives given by (8.1.13) yields

$$\sum_n[a_n p_1\hat{n}\cdot\nabla\Re e\psi_n(k_1,\mathbf{r}) + f_n p_1\hat{n}\cdot\nabla\psi_n(k_1,\mathbf{r})]$$
$$= \sum_n \alpha_n p_2\hat{n}\cdot\nabla\Re e\psi_n(k_2,\mathbf{r}), \quad \mathbf{r}\in S. \tag{8.5.4}$$

To convert the above into matrix equations, we test Equation (3) by $\hat{n}\cdot\nabla\Re e\psi_m(k_2,\mathbf{r})$ and integrate over S to yield

$$\sum_n\left[a_n\int_S dS\,\hat{n}\cdot\nabla\Re e\psi_m(k_2,\mathbf{r})\Re e\psi_n(k_1,\mathbf{r}) + f_n\int_S dS\,\hat{n}\cdot\nabla\Re e\psi_m(k_2,\mathbf{r})\psi_n(k_1,\mathbf{r})\right]$$
$$= \sum_n \alpha_n\int_S dS\,\hat{n}\cdot\nabla\Re e\psi_m(k_2,\mathbf{r})\Re e\psi_n(k_2,\mathbf{r}). \tag{8.5.5}$$

Similarly, we test Equation (4) by $\Re e\psi_m(k_2,\mathbf{r})$ and integrate over S to yield

$$\sum_n \left[a_n \frac{p_1}{p_2} \int_S dS\, \Re e\psi_m(k_2,\mathbf{r})\,\hat{n}\cdot\nabla\Re e\psi_n(k_1,\mathbf{r}) + f_n \frac{p_1}{p_2}\int_S dS\,\Re e\psi_m(k_2,\mathbf{r})\,\hat{n}\cdot\nabla\psi_n(k_1,\mathbf{r})\right] = \sum_n \alpha_n \int_S dS\,\Re e\psi_m(k_2,\mathbf{r})\,\hat{n}\cdot\nabla\Re e\psi_n(k_2,\mathbf{r}). \tag{8.5.6}$$

But the right-hand sides of (5) and (6) are equal as a result of (8.3.13). Consequently, we have

$$\sum_n a_n \left[\frac{p_1}{p_2}\int_S dS\,\Re e\psi_m(k_2,\mathbf{r})\hat{n}\cdot\nabla\Re e\psi_n(k_1,\mathbf{r}) - \int_S dS\,\hat{n}\cdot\nabla\Re e\psi_m(k_2,\mathbf{r})\Re e\psi_n(k_1,\mathbf{r})\right]$$
$$= -\sum_n f_n\left[\frac{p_1}{p_2}\int_S dS\,\Re e\psi_m(k_2,\mathbf{r})\,\hat{n}\cdot\nabla\psi_n(k_1,\mathbf{r}) - \int_S dS\,\hat{n}\cdot\nabla\Re e\psi_m(k_2,\mathbf{r})\psi_n(k_1,\mathbf{r})\right]. \tag{8.5.7}$$

Note that the above is the same as

$$\sum_n a_n \Re e Q_{nm} = -\sum f_n Q_{nm}, \tag{8.5.8}$$

which is the same as

$$\Re e\overline{\mathbf{Q}}^t\cdot\mathbf{a} = -\overline{\mathbf{Q}}^t\cdot\mathbf{f}. \tag{8.5.9}$$

Consequently,

$$\mathbf{f} = -\left(\overline{\mathbf{Q}}^t\right)^{-1}\cdot\Re e\overline{\mathbf{Q}}^t\cdot\mathbf{a}, \tag{8.5.10}$$

or the $\overline{\mathbf{T}}$ matrix is

$$\overline{\mathbf{T}} = -\left(\overline{\mathbf{Q}}^t\right)^{-1}\cdot\Re e\overline{\mathbf{Q}}^t. \tag{8.5.11}$$

Observe that Equation (11) is exactly the transpose of Equation (8.4.6) derived by the EBC method. But from the reciprocity condition (8.4.13), the actual $\overline{\mathbf{T}}$ is a symmetric matrix. Therefore, the $\overline{\mathbf{T}}$ matrix derived with

Rayleigh's hypothesis has formally the same error as that derived by the extended-boundary-condition method, even when the $\overline{\mathbf{Q}}$ matrices are truncated (see Exercise 8.27).

The equivalence of EBC and Rayleigh's method has led to much confusion and controversy in the past (Burrow 1969; Millar 1969; Lewin 1970). In particular, the equivalence of Rayleigh's method to the seemingly more rigorous extended-boundary-condition method has been used to establish its legitimacy. However, it is easy to find counterexamples to Rayleigh's hypothesis in the high frequency limit. In this limit, a bouncing ray picture of the waves clearly indicates the existence of incoming waves as well in V_1. Hence, Rayleigh's method is sometimes an approximate method, as is the EBC method as noted earlier.

The EBC method of imposing the extinction theorem on S_1 and S_2 does not imply the extinction of the field everywhere in V_1 and V_2 respectively. However, the exact solution of the surface integral equation yields surface sources that extinct the appropriate field everywhere in V_1 and V_2. This explains the equivalence of these two methods and the same degree of errors in both the solutions. Consequently, the ill-conditioning of the matrix in Rayleigh's method, which gives rise to poor results, is also due to the presence of localized waves or evanescent waves for a highly corrugated or elongated object. Despite its shortfall, Rayleigh' method is attractive because of the simplicity of its derivation compared to the EBC method. [It has been proven for certain surfaces that Equation (1b) does not converge on S (see references in van den Berg 1980).[12]]

§8.6 Scattering by Many Scatterers

Once the $\overline{\mathbf{T}}$ matrix for one scatterer is found, it can be used easily to construct the solution of scattering by many scatterers. But when more than one scatterer is present, there exists multiple scattering between the scatterers. Nonetheless, by applying the translational addition theorem for spherical harmonics or cylindrical harmonics, the solution to such a problem is easily found. In this section, we shall consider first the solution of two scatterers. Then, we shall derive a recursive algorithm for the solution of N scatterers. The N scatterer solution has been presented by Peterson and Ström (1973, 1974a), but the solution we present here will be in a different light (Chew 1989; Chew and Wang 1990; Chew et al. 1990; Wang and Chew 1990; also see Kerker 1988).

§§*8.6.1 Two-Scatterer Solution*

When two scatterers are present as shown in Figure 8.6.1, we can expand the incident field as

$$\phi_{inc}(\mathbf{r}) = \Re e\psi^t(k_0,\mathbf{r}_0)\cdot\mathbf{a}, \tag{8.6.1}$$

[12] The Rayleigh's hypothesis method has been modified by various scientists, a review of which is given by van den Berg (1980).

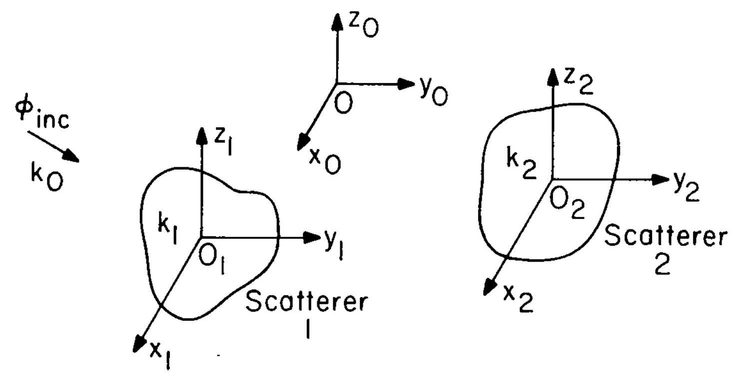


Figure 8.6.1 Two scatterers in the presence of an incident field.

while the scattered field, as

$$\phi_{sca}(\mathbf{r}) = \psi^t(k_0, \mathbf{r}_1) \cdot \mathbf{f}_1 + \psi^t(k_0, \mathbf{r}_2) \cdot \mathbf{f}_2. \tag{8.6.2}$$

Notice that in the above, the scattered wave from each scatterer is expanded in terms of the outgoing harmonics expressed in the self-coordinates of the scatterers. Fortunately enough, translational formulas exist both for cylindrical and spherical harmonics such that

$$\psi^t(k_0, \mathbf{r}_i) = \Re e\psi^t(k_0, \mathbf{r}_j) \cdot \overline{\boldsymbol{\alpha}}_{ji}, \qquad |\mathbf{r}_j| < d_{ij}, \tag{8.6.3a}$$

$$\psi^t(k_0, \mathbf{r}_i) = \psi^t(k_0, \mathbf{r}_j) \cdot \overline{\boldsymbol{\beta}}_{ji}, \qquad |\mathbf{r}_j| > d_{ij}, \tag{8.6.3b}$$

$$\Re e\psi^t(k_0, \mathbf{r}_i) = \Re e\psi^t(k_0, \mathbf{r}_j) \cdot \overline{\boldsymbol{\beta}}_{ji}, \qquad \forall\, |\mathbf{r}_j|, \tag{8.6.3c}$$

where d_{ij} is the distance between the O_i and O_j, the origins of the i and j coordinates (Friedman and Russek 1954; Stein 1961; Cruzan 1962; Danos and Maximon 1965; Chew 1989; and Chew et al. 1990; also see Appendix D). These formulas allow expression of the harmonic expansion of the field in one coordinate system in terms of another coordinate system readily. In general, $\overline{\boldsymbol{\beta}}_{ji} = \Re e\overline{\boldsymbol{\alpha}}_{ji}$ where $\Re e$ stands for "the regular part of" (see Exercise 8.28).

Then, using (3a) and (3c), the total field exterior to the scatterers expressed in terms of the coordinates of the first scatterer is

$$\phi(\mathbf{r}) = \Re e\psi^t(k_0, \mathbf{r}_1) \cdot \overline{\boldsymbol{\beta}}_{10} \cdot \mathbf{a} + \psi^t(k_0, \mathbf{r}_1) \cdot \mathbf{f}_1 + \Re e\psi^t(k_0, \mathbf{r}_1) \cdot \overline{\boldsymbol{\alpha}}_{12} \cdot \mathbf{f}_2. \tag{8.6.4}$$

The first and the third terms in the above can be viewed as the incident field impinging on the scatterer 1, while the second term is the scattered field from

scatterer 1. But if the $\overline{\mathbf{T}}$ matrix of the first scatterer when it is isolated is known, we can write a relationship between $\mathbf{a}$, $\mathbf{f}_1$, and $\mathbf{f}_2$ using this $\overline{\mathbf{T}}$ matrix. In other words,

$$\mathbf{f}_1 = \overline{\mathbf{T}}_{1(1)} \cdot \left[\overline{\boldsymbol{\beta}}_{10} \cdot \mathbf{a} + \overline{\boldsymbol{\alpha}}_{12} \cdot \mathbf{f}_2\right]. \tag{8.6.5}$$

Similarly, for scatterer 2, we have

$$\mathbf{f}_2 = \overline{\mathbf{T}}_{2(1)} \cdot \left[\overline{\boldsymbol{\beta}}_{20} \cdot \mathbf{a} + \overline{\boldsymbol{\alpha}}_{21} \cdot \mathbf{f}_1\right]. \tag{8.6.6}$$

In the above, $\overline{\mathbf{T}}_{i(1)}$ is the isolated-scatterer $\overline{\mathbf{T}}$ matrix for the i-th scatterer; the parenthesized 1 indicates that it is the one-scatterer $\overline{\mathbf{T}}$ matrix.

Equations (5) and (6) can be solved to yield

$$\mathbf{f}_1 = \left[\overline{\mathbf{I}} - \overline{\mathbf{T}}_{1(1)} \cdot \overline{\boldsymbol{\alpha}}_{12} \cdot \overline{\mathbf{T}}_{2(1)} \cdot \overline{\boldsymbol{\alpha}}_{21}\right]^{-1} \cdot \overline{\mathbf{T}}_{1(1)} \cdot \left[\overline{\boldsymbol{\beta}}_{10} + \overline{\boldsymbol{\alpha}}_{12} \cdot \overline{\mathbf{T}}_{2(1)} \cdot \overline{\boldsymbol{\beta}}_{20}\right] \cdot \mathbf{a}, \tag{8.6.7}$$

$$\mathbf{f}_2 = \left[\overline{\mathbf{I}} - \overline{\mathbf{T}}_{2(1)} \cdot \overline{\boldsymbol{\alpha}}_{21} \cdot \overline{\mathbf{T}}_{1(1)} \cdot \overline{\boldsymbol{\alpha}}_{12}\right]^{-1} \cdot \overline{\mathbf{T}}_{2(1)} \cdot \left[\overline{\boldsymbol{\beta}}_{20} + \overline{\boldsymbol{\alpha}}_{21} \cdot \overline{\mathbf{T}}_{1(1)} \cdot \overline{\boldsymbol{\beta}}_{10}\right] \cdot \mathbf{a}. \tag{8.6.8}$$

Now, from (7) and (8), new $\overline{\mathbf{T}}$ matrices are defined such that

$$\mathbf{f}_1 = \overline{\mathbf{T}}_{1(2)} \cdot \overline{\boldsymbol{\beta}}_{10} \cdot \mathbf{a}, \tag{8.6.9}$$

$$\mathbf{f}_2 = \overline{\mathbf{T}}_{2(2)} \cdot \overline{\boldsymbol{\beta}}_{20} \cdot \mathbf{a}, \tag{8.6.10}$$

where now, $\overline{\mathbf{T}}_{i(2)}$ is a two-scatterer $\overline{\mathbf{T}}$ matrix. It relates the total scattered field due to the i-th scatterer to the incident field amplitude when two scatterers are present. Notice that the equations for $\overline{\mathbf{T}}_{i(2)}$ can be derived by comparing (9) and (10) with (7) and (8). Moreover, the factor $\overline{\boldsymbol{\beta}}_{i0}$ is introduced so that the $\overline{\mathbf{T}}$ matrices are still defined with respect to the self-coordinates of the scatterers.

§§8.6.2 *N-Scatterer Solution—A Recursive Algorithm*

The previous subsection illustrated how the two-scatterer solution can be constructed from the one-scatterer solution. This concept can be further extended to find the scattering solution of $n+1$ scatterers given the solution of the scattering from n scatterers (see Figure 8.6.2). Now, if we define an n-scatterer $\overline{\mathbf{T}}$ matrix $\overline{\mathbf{T}}_{i(n)}$, then the total field external to the n scatterers is of the form

$$\phi(\mathbf{r}) = \Re e\psi^t(k_0, \mathbf{r}_0) \cdot \mathbf{a} + \sum_{i=1}^{n} \psi^t(k_0, \mathbf{r}_i) \cdot \overline{\mathbf{T}}_{i(n)} \cdot \overline{\boldsymbol{\beta}}_{i0} \cdot \mathbf{a}. \tag{8.6.11}$$

Similarly, the $(n+1)$-scatterer solution has the form

$$\phi(\mathbf{r}) = \Re e\psi^t(k_0, \mathbf{r}_0) \cdot \mathbf{a} + \sum_{i=1}^{n+1} \psi^t(k_0, \mathbf{r}_i) \cdot \overline{\mathbf{T}}_{i(n+1)} \cdot \overline{\boldsymbol{\beta}}_{i0} \cdot \mathbf{a}. \tag{8.6.12}$$

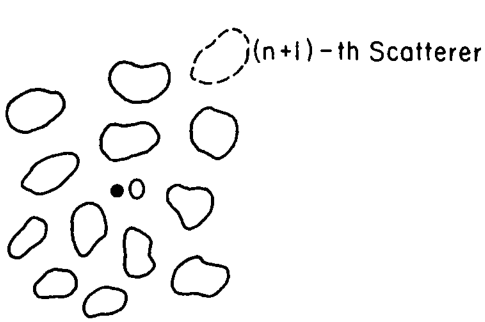


Figure 8.6.2 A recursive algorithm can be derived such that the scattering solution from $n+1$ scatterers can be obtained from the scattering solution of n scatterers.

The preceding equation can be written more suggestively as

$$\phi(\mathbf{r}) = \Re e\psi^t(k_0, \mathbf{r}_0) \cdot \mathbf{a} + \sum_{i=1}^{n} \psi^t(k_0, \mathbf{r}_i) \cdot \overline{\mathbf{T}}_{i(n+1)} \cdot \overline{\boldsymbol{\beta}}_{i0} \cdot \mathbf{a} + \psi^t(k_0, \mathbf{r}_{n+1}) \cdot \overline{\mathbf{T}}_{n+1(n+1)} \cdot \overline{\boldsymbol{\beta}}_{n+1,0} \cdot \mathbf{a}. \quad (8.6.13)$$

Now, the first and the last terms in Equation (13) can be thought of as incident fields impinging on the $i = 1, \ldots, n$ scatterers. Therefore,

$$\overline{\mathbf{T}}_{i(n+1)} \cdot \overline{\boldsymbol{\beta}}_{i0} \cdot \mathbf{a} = \overline{\mathbf{T}}_{i(n)} \cdot \overline{\boldsymbol{\beta}}_{i0} \cdot \left[\overline{\boldsymbol{I}} + \overline{\boldsymbol{\alpha}}_{0,n+1} \cdot \overline{\mathbf{T}}_{n+1(n+1)} \cdot \overline{\boldsymbol{\beta}}_{n+1,0}\right] \cdot \mathbf{a}, \quad (8.6.14)$$

where we have used the translation formulas (3) to translate the last terms to the global coordinates $\mathbf{r}_0$. But since $\overline{\mathbf{T}}_{i(n)} \cdot \overline{\boldsymbol{\beta}}_{io}$ is defined only for a source outside the smallest circle centered at $\mathbf{r}_0 = 0$ and circumscribing the n spheres, the $(n+1)$-th scatterer must be on or outside this circle. In other words, the distances of the scatterers from $\mathbf{r}_0 = 0$ have to be ordered.

Furthermore, the scattered field amplitude from the $(n+1)$-th scatterer is due to the scattering of the incident field from the other n scatterers via the isolated-scatterer $\overline{\mathbf{T}}$ matrix. Hence, the scattered field amplitude due to the $(n+1)$-th scatterer is related to the other field amplitudes as

$$\overline{\mathbf{T}}_{n+1(n+1)} \cdot \overline{\boldsymbol{\beta}}_{n+1,0} \cdot \mathbf{a} = \overline{\mathbf{T}}_{n+1(1)} \cdot \left[\overline{\boldsymbol{\beta}}_{n+1,0} + \sum_{i=1}^{n} \overline{\boldsymbol{\alpha}}_{n+1,i} \cdot \overline{\mathbf{T}}_{i(n+1)} \cdot \overline{\boldsymbol{\beta}}_{i0}\right] \cdot \mathbf{a}. \quad (8.6.15)$$

Consequently, using (14) in (15), we have

$$\overline{\mathbf{T}}_{n+1(n+1)} \cdot \overline{\boldsymbol{\beta}}_{n+1,0} = \overline{\mathbf{T}}_{n+1(1)} \cdot \left[\overline{\boldsymbol{\beta}}_{n+1,0} + \sum_{i=1}^{n} \overline{\boldsymbol{\alpha}}_{n+1,i} \cdot \overline{\mathbf{T}}_{i(n)} \cdot \overline{\boldsymbol{\beta}}_{i0} + \sum_{i=1}^{n} \overline{\boldsymbol{\alpha}}_{n+1,i} \cdot \overline{\mathbf{T}}_{i(n)} \cdot \overline{\boldsymbol{\beta}}_{io} \cdot \overline{\boldsymbol{\alpha}}_{0,n+1} \cdot \overline{\mathbf{T}}_{n+1(n+1)} \cdot \overline{\boldsymbol{\beta}}_{n+1,0}\right]. \quad (8.6.16)$$

Then, after solving this equation for $\overline{\mathbf{T}}_{n+1(n+1)} \cdot \overline{\boldsymbol{\beta}}_{n+1,0}$, we obtain

$$\overline{\mathbf{T}}_{n+1(n+1)} \cdot \overline{\boldsymbol{\beta}}_{n+1,0} = \left[\overline{\mathbf{I}} - \overline{\mathbf{T}}_{n+1(1)} \cdot \sum_{i=1}^{n} \overline{\boldsymbol{\alpha}}_{n+1,i} \cdot \overline{\mathbf{T}}_{i(n)} \cdot \overline{\boldsymbol{\beta}}_{io} \cdot \overline{\boldsymbol{\alpha}}_{0,n+1}\right]^{-1} \cdot \overline{\mathbf{T}}_{n+1(1)} \cdot \left[\overline{\boldsymbol{\beta}}_{n+1,0} + \sum_{i=1}^{n} \overline{\boldsymbol{\alpha}}_{n+1,i} \cdot \overline{\mathbf{T}}_{i(n)} \cdot \overline{\boldsymbol{\beta}}_{i0}\right]. \quad (8.6.17)$$

But from (14), we have

$$\overline{\mathbf{T}}_{i(n+1)} \cdot \overline{\boldsymbol{\beta}}_{i0} = \overline{\mathbf{T}}_{i(n)} \cdot \overline{\boldsymbol{\beta}}_{i0} \cdot [\overline{\mathbf{I}} + \overline{\boldsymbol{\alpha}}_{0,n+1} \cdot \overline{\mathbf{T}}_{n+1(n+1)} \cdot \overline{\boldsymbol{\beta}}_{n+1,0}]. \quad (8.6.18)$$

Therefore, Equations (17) and (18) together constitute the recursive relations enabling one to calculate the $\overline{\mathbf{T}}_{i(n+1)} \cdot \overline{\boldsymbol{\beta}}_{io}$ matrices, $i = 1, \ldots, n+1$, given the $\overline{\mathbf{T}}_{i(n)} \cdot \overline{\boldsymbol{\beta}}_{io}$ matrices, $i = 1, \ldots, n$. Therefore, given the knowledge of the isolated-scatterer $\overline{\mathbf{T}}$ matrices, the N-scatterer solution is constructed recursively, starting from the one-scatterer solution. In this manner, only small matrices determined by the dimensions of the $\overline{\mathbf{T}}$ matrices must be dealt with at each recursion. Consequently, only a small amount of computer memory is required at each recursion, which reduces the number of page-faults in a virtual memory machine.

In the above, if there are N scatterers, and the field around each scatterer is approximated by M harmonics, there are altogether NM unknowns. In this case, $\overline{\mathbf{T}}_{i(n)} \cdot \overline{\boldsymbol{\beta}}_{i0} \cdot \mathbf{a}$ in (11) is an M element column vector. But when the scatterers are small, M, the number of unknowns on each scatterer, can be kept small. On the contrary, the number of terms in the translation formulas should be large enough to maintain their accuracy. In other words, $\overline{\mathbf{T}}_{i(n)} \cdot \overline{\boldsymbol{\beta}}_{i0}$ need not be square—it should be a $M \times P$ matrix where P is large enough to keep the translation accurate.

In view of this, the dimensions of the matrices in (17) and (18) are indicated as

$$\overbrace{\overline{\mathbf{T}}_{n+1(n+1)} \cdot \overline{\boldsymbol{\beta}}_{n+1,0}}^{M\times P} = \left[\overline{\mathbf{I}} - \overbrace{\overline{\mathbf{T}}_{n+1(1)}}^{M\times M} \cdot \sum_{i=1}^{n} \overbrace{\overline{\boldsymbol{\alpha}}_{n+1,i}}^{M\times M} \cdot \overbrace{\overline{\mathbf{T}}_{i(n)} \cdot \overline{\boldsymbol{\beta}}_{i0}}^{M\times P} \cdot \overbrace{\overline{\boldsymbol{\alpha}}_{0,n+1}}^{P\times M}\right]^{-1} \cdot \underbrace{\overline{\mathbf{T}}_{n+1(1)}}_{M\times M} \cdot \left[\underbrace{\overline{\boldsymbol{\beta}}_{n+1,0}}_{M\times P} + \sum_{i=1}^{n} \underbrace{\overline{\boldsymbol{\alpha}}_{n+1,i}}_{M\times M} \cdot \underbrace{\overline{\mathbf{T}}_{i(n)} \cdot \overline{\boldsymbol{\beta}}_{i0}}_{M\times P}\right], \quad (8.6.19)$$

$$\underbrace{\overline{\mathbf{T}}_{i(n+1)}\cdot\overline{\boldsymbol{\beta}}_{i0}}_{M\times P} = \underbrace{\overline{\mathbf{T}}_{i(n)}\cdot\overline{\boldsymbol{\beta}}_{i0}}_{M\times P} + (\underbrace{\overline{\mathbf{T}}_{i(n)}\cdot\overline{\boldsymbol{\beta}}_{i0}}_{M\times P}\cdot\underbrace{\overline{\boldsymbol{\alpha}}_{0,n+1}}_{P\times M})\cdot\underbrace{\overline{\mathbf{T}}_{n+1(n+1)}\cdot\overline{\boldsymbol{\beta}}_{n+1,0}}_{M\times P}\cdot \tag{8.6.20}$$

In the above, $\overline{\mathbf{T}}_{i(n+1)}\cdot\overline{\boldsymbol{\beta}}_{i0}$ can be regarded as the function to be solved for; it is an $M\times P$ matrix. (The dimensions of the matrices are indicated in the equations above.) Notice that the number of floating point operations required to multiply an $M\times P$ matrix with a $P\times M$ matrix, or an $M\times M$ matrix with an $M\times P$ matrix is equal to M^2P. But since $M \ll P$, the other matrix multiplications and inversions are subdominant. Therefore, at each recursion, the number of floating point operation is $O(nM^2P)$ after counting the dominant matrix multiplications in (19) and (20). Consequently, after applying the recursion relations to N scatterers, the number of cumulative floating point operation is $O(N^2M^2P)$ (Exercise 8.29).

The N-scatterer problem is also expressible as an NM unknown problem by solving NM linear algebraic equations. This would require $O(N^3M^3)$ floating-point operations, however, if these NM linear algebraic equations are solved with Gauss' elimination. On the other hand, if the conjugate gradient method is used here, $O(N^{2+\alpha})$ algorithm (where α depends on the condition number of the matrix) is possible. But still, the conjugate gradient method is an iterative procedure that solves the matrix equation $\overline{\mathbf{A}}\cdot\mathbf{x}=\mathbf{b}$ with a fixed right-hand side. Therefore, it has to be restarted if the right-hand side of the equation changes, and if the incident angle of the wave changes, the equation needs to be solved again. However, the preceding algorithm derived is independent of the incident angle of the incident wave. The reduction in computational effort here can be traced to the fact that the $\overline{\boldsymbol{\alpha}}_{ij}$ or $\overline{\boldsymbol{\beta}}_{ij}$ matrices are the representation of a translation group.

The recursive relations given by (19) and (20) can be further manipulated to a different form by letting $\overline{\boldsymbol{\alpha}}_{n+1,i}=\overline{\boldsymbol{\alpha}}_{n+1,0}\cdot\overline{\boldsymbol{\beta}}_{0i}$. Then, (19) becomes

$$\overline{\mathbf{T}}_{n+1(n+1)}\cdot\overline{\boldsymbol{\beta}}_{n+1,0} = \left[\overline{\mathbf{I}}-\overline{\mathbf{T}}_{n+1(1)}\cdot\overline{\boldsymbol{\alpha}}_{n+1,0}\cdot\left(\sum_{i=1}^{n}\overline{\boldsymbol{\beta}}_{0i}\cdot\overline{\mathbf{T}}_{i(n)}\cdot\overline{\boldsymbol{\beta}}_{i0}\right)\cdot\overline{\boldsymbol{\alpha}}_{0,n+1}\right]^{-1}$$
$$\cdot\,\overline{\mathbf{T}}_{n+1(1)}\cdot\left[\overline{\boldsymbol{\beta}}_{n+1,0}+\overline{\boldsymbol{\alpha}}_{n+1,0}\cdot\left(\sum_{i=1}^{n}\overline{\boldsymbol{\beta}}_{0i}\cdot\overline{\mathbf{T}}_{i(n)}\cdot\overline{\boldsymbol{\beta}}_{i0}\right)\right]. \tag{8.6.21}$$

Then, an aggregate $\overline{\mathbf{T}}$ matrix for n scatterers can be defined such that

$$\overline{\boldsymbol{\tau}}_{(n)}=\sum_{i=1}^{n}\overline{\boldsymbol{\beta}}_{0i}\cdot\overline{\mathbf{T}}_{i(n)}\cdot\overline{\boldsymbol{\beta}}_{i0}. \tag{8.6.22}$$

And (21) becomes

$$\overline{\mathbf{T}}_{n+1(n+1)}\cdot\overline{\boldsymbol{\beta}}_{n+1,0}=\left[\overline{\mathbf{I}}-\overline{\mathbf{T}}_{n+1(1)}\cdot\overline{\boldsymbol{\alpha}}_{n+1,0}\cdot\overline{\boldsymbol{\tau}}_{(n)}\cdot\overline{\boldsymbol{\alpha}}_{0,n+1}\right]^{-1}$$
$$\cdot\,\overline{\mathbf{T}}_{n+1(1)}\cdot\left[\overline{\boldsymbol{\beta}}_{n+1,0}+\overline{\boldsymbol{\alpha}}_{n+1,0}\cdot\overline{\boldsymbol{\tau}}_{(n)}\right]. \tag{8.6.23}$$

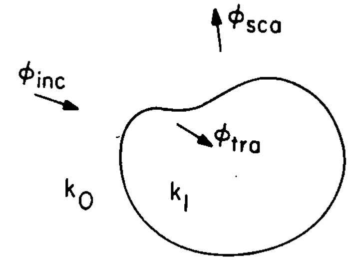


Figure 8.7.1 The one-interface problem for defining the $\overline{\mathbf{T}}$ matrices for the incoming wave case.

Moreover, on multiplying (20) by $\overline{\boldsymbol{\beta}}_{0i}$ and summing over i from 1 to n, we have

$$\overline{\boldsymbol{\tau}}_{(n+1)}=\overline{\boldsymbol{\tau}}_{(n)}+\left[\overline{\boldsymbol{\beta}}_{0,n+1}+\overline{\boldsymbol{\tau}}_{(n)}\cdot\overline{\boldsymbol{\alpha}}_{0,n+1}\right]\cdot\overline{\mathbf{T}}_{n+1(n+1)}\cdot\overline{\boldsymbol{\beta}}_{n+1,0}. \tag{8.6.24}$$

Now, Equations (23) and (24) constitute the recursion relations expressing $\overline{\boldsymbol{\tau}}_{(n+1)}$ in terms of $\overline{\boldsymbol{\tau}}_{(n)}$. Furthermore, when M multipoles are assumed for each scatterer and P harmonics used for the translation formulas, a count shows that the above is an NMP^2 algorithm. Consequently, if M and P could be kept small, this is a very efficient method of calculating the scattering from many scatterers when N is large. Moreover, an arbitrary shape, inhomogeneous scatterer can be divided into N subscatterers, and its scattering solved by such an algorithm. (The above concepts are easily adapted to the vector electromagnetic scattering problems.)

§8.7 Scattering by Multilayered Scatterers

Thus far, how the $\overline{\mathbf{T}}$ matrix of a single scatterer and many scatterers could be derived recursively has been shown. This concept can be extended to the case of a multilayered scatterer. To do this, it is expedient to elucidate the physics of the scattering at each interface. So, first we shall derive the $\overline{\mathbf{T}}$ matrices for the one interface problem and, ultimately, derive the multi-interface problem from it. Although this solution was originally presented by Peterson and Ström (1974b, 1975) and Wang and Barber (1979), the solution here is considered in a more general sense.

§§*8.7.1 One-Interface Problem*

For the geometry shown in Figure 8.7.1, the field external to the scatterer is of the form

$$\phi(\mathbf{r})=\Re e\boldsymbol{\psi}^t(k_0,\mathbf{r})\cdot\mathbf{a}+\boldsymbol{\psi}^t(k_0,\mathbf{r})\cdot\overline{\mathbf{R}}_{01}\cdot\mathbf{a}, \tag{8.7.1}$$

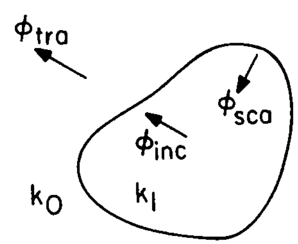


Figure 8.7.2 The one-interface problem for defining the $\overline{\mathbf{T}}$ matrices for the outgoing wave case.

where $\overline{\mathbf{R}}_{01}$ is a matrix equivalent to the $\overline{\mathbf{T}}$ matrix previously defined. In keeping with the spirit of Chapter 3, it is called the reflection matrix here, and $\overline{\mathbf{T}}$ is reserved for a transmission matrix in this section. In addition, the field internal to the scatterer is expressible as

$$\phi_{tra} = \Re e \psi^t(k_1, \mathbf{r}) \cdot \boldsymbol{\alpha}, \tag{8.7.2}$$

where $\boldsymbol{\alpha}$ is a column vector containing α_m's. This internal field is solvable via the EBC method from (8.3.15), which implies that

$$i\overline{\mathbf{Q}} \cdot \boldsymbol{\alpha} = \mathbf{a}, \quad \text{or} \quad \boldsymbol{\alpha} = -i\overline{\mathbf{Q}}^{-1} \cdot \mathbf{a}. \tag{8.7.3}$$

Next, we define a transmission matrix such that

$$\phi_{tra} = \Re e \psi^t(k_1, \mathbf{r}) \cdot \overline{\mathbf{T}}_{01} \cdot \mathbf{a}, \tag{8.7.4}$$

where

$$\overline{\mathbf{T}}_{01} = -i\overline{\mathbf{Q}}^{-1}. \tag{8.7.4a}$$

Now, consider the case where the field is incident at the interface from the inside, as shown in Figure 8.7.2. The incident wave in this case is the outgoing wave. Then, the field internal to the scatterer is of the form

$$\phi(\mathbf{r}) = \phi_{inc}(\mathbf{r}) + \phi_{sca}(\mathbf{r}) = \psi^t(k_1, \mathbf{r}) \cdot \mathbf{a} + \Re e \psi^t(k_1, \mathbf{r}) \cdot \mathbf{f}. \tag{8.7.5}$$

And the field external to the scatterer is

$$\phi_{tra}(\mathbf{r}) = \psi^t(k_0, \mathbf{r}) \cdot \boldsymbol{\alpha}. \tag{8.7.6}$$

The above problem is again solvable by the EBC method (see Exercise 8.30). Therefore, the waves are

$$\phi(\mathbf{r}) = \psi^t(k_1, \mathbf{r}) \cdot \mathbf{a} + \Re e \psi^t(k_1, \mathbf{r}) \cdot \overline{\mathbf{R}}_{10} \cdot \mathbf{a}, \qquad \mathbf{r} \in \text{region } 1, \tag{8.7.7a}$$

$$\phi_{tra}(\mathbf{r}) = \psi^t(k_0, \mathbf{r}) \cdot \overline{\mathbf{T}}_{10} \cdot \mathbf{a}, \qquad \mathbf{r} \in \text{region } 0. \tag{8.7.7b}$$

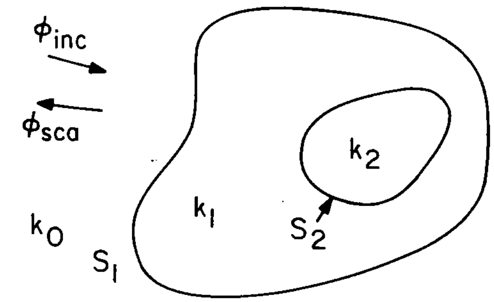


Figure 8.7.3 The two-interface problem.

Consequently, with the canonical problems defined for the one-interface problem, the solution for the many-interface problems is easily derived as shown in the following.

§§8.7.2 *Many-Interface Problem*

Consider the two-interface problem shown in Figure 8.7.3; the two surfaces S_1 and S_2 may not be concentric. So, it is necessary to express the $\overline{\mathbf{T}}$ matrices of the two interfaces in different coordinate systems. This is easily achieved via the translation formulas described in the previous section.

Consequently, in region 0, the field is

$$\phi_0(\mathbf{r}) = \Re e \psi^t(k_0, \mathbf{r}_1) \cdot \mathbf{a}_0 + \psi^t(k_0, \mathbf{r}_1) \cdot \mathbf{b}_0; \tag{8.7.8}$$

in region 1, the field is

$$\phi_1(\mathbf{r}) = \Re e \psi^t(k_1, \mathbf{r}_1) \cdot \mathbf{a}_1 + \psi^t(k_1, \mathbf{r}_2) \cdot \mathbf{b}_1; \tag{8.7.9}$$

and in region 2, the field is

$$\phi_2(\mathbf{r}) = \Re e \psi^t(k_2, \mathbf{r}_2) \cdot \mathbf{a}_2, \tag{8.7.10}$$

where $\mathbf{r}_1$ is in the coordinates for the first interface S_1, and $\mathbf{r}_2$ is in the coordinates for the second interface S_2.[13] Hence, the scattered fields, which are the second terms in (8) and (9), are expressed in the coordinates of the surfaces that cause the scattering. The transmitted field in (10) is also expressed in the coordinates of the surface that causes the transmission.

[13] The coordinates for a surface S should be chosen so that the inscribed and exscribed spheres shown in Figure 8.3.1 are not too far from the surface S in order to avoid ill-conditioned $\overline{\mathbf{T}}$ matrices.

By requiring the outgoing wave in region 0 to be a consequence of the reflection of the incident wave plus the transmission of the outgoing wave in region 1, we have

$$\mathbf{b}_0 = \overline{\mathbf{R}}_{01} \cdot \mathbf{a}_0 + \overline{\mathbf{T}}_{10} \cdot \overline{\boldsymbol{\beta}}_{12} \cdot \mathbf{b}_1. \qquad (8.7.11)$$

In the above, the translation formula [see (8.6.3)]

$$\boldsymbol{\psi}^t(k_1, \mathbf{r}_2) \cdot \mathbf{b}_1 = \boldsymbol{\psi}^t(k_1, \mathbf{r}_1) \cdot \overline{\boldsymbol{\beta}}_{12} \cdot \mathbf{b}_1 \qquad (8.7.12)$$

is used to translate the outgoing wave in the $\mathbf{r}_2$ coordinates to the $\mathbf{r}_1$ coordinates. The incoming wave in region 1, $\Re e\boldsymbol{\psi}^t(k_1, \mathbf{r}_1) \cdot \mathbf{a}_1$, is a consequence of the transmission of the incoming wave in region 0 plus the reflection of the outgoing wave in region 1. Therefore,

$$\mathbf{a}_1 = \overline{\mathbf{T}}_{01} \cdot \mathbf{a}_0 + \overline{\mathbf{R}}_{10} \cdot \overline{\boldsymbol{\beta}}_{12} \cdot \mathbf{b}_1. \qquad (8.7.13)$$

By the same token, we express

$$\Re e\boldsymbol{\psi}^t(k_1, \mathbf{r}_1) \cdot \mathbf{a}_1 = \Re e\boldsymbol{\psi}^t(k_1, \mathbf{r}_2) \cdot \overline{\boldsymbol{\beta}}_{21} \cdot \mathbf{a}_1 \qquad (8.7.14)$$

so that (9) becomes

$$\phi_1(\mathbf{r}) = \Re e\boldsymbol{\psi}^t(k_1, \mathbf{r}_2) \cdot \overline{\boldsymbol{\beta}}_{21} \cdot \mathbf{a}_1 + \boldsymbol{\psi}^t(k_1, \mathbf{r}_2) \cdot \mathbf{b}_1. \qquad (8.7.15)$$

Then,

$$\mathbf{b}_1 = \overline{\mathbf{R}}_{12} \cdot \overline{\boldsymbol{\beta}}_{21} \cdot \mathbf{a}_1, \qquad (8.7.16)$$

where $\overline{\mathbf{R}}_{12}$ is the reflection matrix for waves incident from region 1 onto the interface between regions 1 and 2.

From (11), (13), and (16), it follows that

$$\mathbf{a}_1 = \overline{\mathbf{M}}_{1-} \cdot \overline{\mathbf{T}}_{01} \cdot \mathbf{a}_0, \qquad (8.7.17)$$

$$\mathbf{b}_1 = \overline{\mathbf{R}}_{12} \cdot \overline{\boldsymbol{\beta}}_{21} \cdot \overline{\mathbf{M}}_{1-} \cdot \overline{\mathbf{T}}_{01} \cdot \mathbf{a}_0, \qquad (8.7.18)$$

and

$$\mathbf{b}_0 = \left[\overline{\mathbf{R}}_{01} + \overline{\mathbf{T}}_{10} \cdot \overline{\boldsymbol{\beta}}_{12} \cdot \overline{\mathbf{R}}_{12} \cdot \overline{\boldsymbol{\beta}}_{21} \cdot \overline{\mathbf{M}}_{1-} \cdot \overline{\mathbf{T}}_{01}\right] \cdot \mathbf{a}_0, \qquad (8.7.19)$$

where $\overline{\mathbf{M}}_{1-} = \left[\overline{\mathbf{I}} - \overline{\mathbf{R}}_{10} \cdot \overline{\boldsymbol{\beta}}_{12} \cdot \overline{\mathbf{R}}_{12} \cdot \overline{\boldsymbol{\beta}}_{21}\right]^{-1}$. Similarly, we deduce that the field in region 2 is the transmission of the field in region 1, and hence,

$$\mathbf{a}_2 = \overline{\mathbf{T}}_{12} \cdot \overline{\boldsymbol{\beta}}_{21} \cdot \mathbf{a}_1 = \overline{\mathbf{T}}_{12} \cdot \overline{\boldsymbol{\beta}}_{21} \cdot \overline{\mathbf{M}}_{1-} \cdot \overline{\mathbf{T}}_{01} \cdot \mathbf{a}_0. \qquad (8.7.20)$$

With these amplitude coefficients known, the field everywhere is found.

Note that from (19), a generalized reflection matrix can be defined for region 0 such that

$$\widetilde{\overline{\mathbf{R}}}_{01} = \overline{\mathbf{R}}_{01} + \overline{\mathbf{T}}_{10} \cdot \overline{\boldsymbol{\beta}}_{12} \cdot \overline{\mathbf{R}}_{12} \cdot \overline{\boldsymbol{\beta}}_{21} \cdot \overline{\mathbf{M}}_{1-} \cdot \overline{\mathbf{T}}_{01}. \qquad (8.7.21)$$

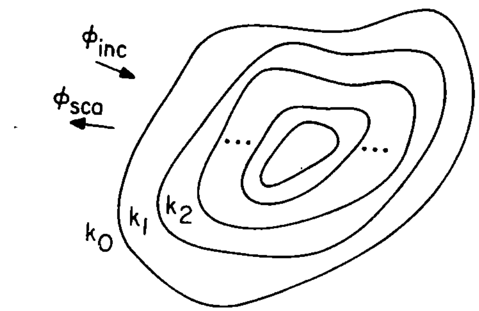


Figure 8.7.4 A multilayered scatterer.

If an inner region is now added to region 2, we need only modify $\overline{\mathbf{R}}_{12}$ to $\widetilde{\overline{\mathbf{R}}}_{12}$ in the above. Consequently, a recursive relation for a many-interface problem as shown in Figure 8.7.4 is expressible as

$$\widetilde{\overline{\mathbf{R}}}_{i,i+1} = \overline{\mathbf{R}}_{i,i+1} + \overline{\mathbf{T}}_{i+1,i} \cdot \overline{\boldsymbol{\beta}}_{i+1,i+2} \cdot \widetilde{\overline{\mathbf{R}}}_{i+1,i+2} \cdot \overline{\boldsymbol{\beta}}_{i+2,i+1} \cdot \overline{\mathbf{M}}_{i+1,-} \cdot \overline{\mathbf{T}}_{i,i+1}, \qquad (8.7.22)$$

where

$$\overline{\mathbf{M}}_{i+1,-} = \left[\overline{\mathbf{I}} - \overline{\mathbf{R}}_{i+1,i} \cdot \overline{\boldsymbol{\beta}}_{i+1,i+2} \cdot \widetilde{\overline{\mathbf{R}}}_{i+1,i+2} \cdot \overline{\boldsymbol{\beta}}_{i+2,i+1}\right]^{-1}, \qquad (8.7.23)$$

and $\widetilde{\overline{\mathbf{R}}}_{i,i+1}$ for the innermost region is zero. Moreover, if the field in region i is written as

$$\phi_i(\mathbf{r}) = \Re e\boldsymbol{\psi}^t(k_i, \mathbf{r}_i) \cdot \mathbf{a}_i + \boldsymbol{\psi}^t(k_i, \mathbf{r}_{i+1}) \cdot \mathbf{b}_i, \qquad (8.7.24)$$

then $\mathbf{a}_i$ could be found recursively as (Exercise 8.31)

$$\mathbf{a}_{i+1} = \overline{\mathbf{M}}_{i+1,-} \cdot \overline{\mathbf{T}}_{i,i+1} \cdot \overline{\boldsymbol{\beta}}_{i+1,i} \cdot \mathbf{a}_i \qquad (8.7.25)$$

with $\mathbf{a}_0$ known. Furthermore, $\mathbf{b}_i$ is related to $\mathbf{a}_i$ as

$$\mathbf{b}_i = \widetilde{\overline{\mathbf{R}}}_{i,i+1} \cdot \overline{\boldsymbol{\beta}}_{i+1,i} \cdot \mathbf{a}_i. \qquad (8.7.26)$$

In this manner, the field everywhere inside the scatterer can be calculated. Note that the translation matrix $\overline{\boldsymbol{\beta}}$ is not necessary if the surfaces are near concentric. However, they are necessary in the example shown in Figure 8.7.3, where it is not possible to expand the $\overline{\mathbf{T}}$ matrices for the two surfaces in one coordinate system. If all the surfaces are concentric circles or spheres, then

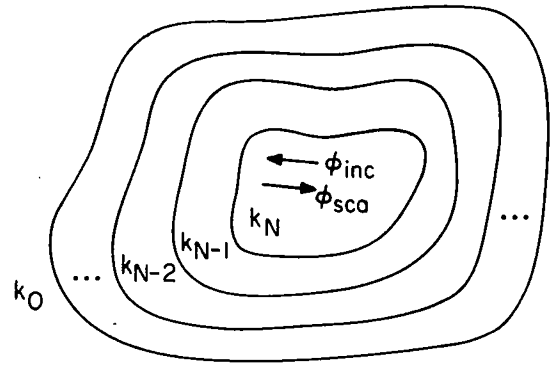


Figure 8.7.5 A multilayered scatterer for the inside-out problem.

the results here reduce to those of Chapter 3. Therefore, we can consider these results a generalization of those in Chapter 3.

On the other hand, if we have an inside-out problem as shown in Figure 8.7.5, the recursive relation for the reflection matrix is

$$\tilde{\overline{\mathbf{R}}}_{i+1,i} = \overline{\mathbf{R}}_{i+1,i} + \overline{\mathbf{T}}_{i,i+1} \cdot \overline{\beta}_{i-1,i} \cdot \tilde{\overline{\mathbf{R}}}_{i,i-1} \cdot \overline{\beta}_{i,i-1} \cdot \overline{\mathbf{M}}_{i,+} \cdot \overline{\mathbf{T}}_{i+1,i}, \tag{8.7.27}$$

where

$$\overline{\mathbf{M}}_{i,+} = \left[\overline{\mathbf{I}} - \tilde{\overline{\mathbf{R}}}_{i,i-1} \cdot \overline{\beta}_{i,i+1} \cdot \overline{\mathbf{R}}_{i,i+1} \cdot \overline{\beta}_{i,i+1}\right]^{-1}, \tag{8.7.28}$$

with $\tilde{\overline{\mathbf{R}}}_{i+1,i}$ for the outermost region being zero. Moreover, if the field in region i is expressed as in (24), then,

$$\mathbf{b}_i = \overline{\mathbf{M}}_{i,+} \cdot \overline{\mathbf{T}}_{i+1,i} \cdot \overline{\beta}_{i,i+1} \cdot \mathbf{b}_{i+1} \tag{8.7.29}$$

allowing all $\mathbf{b}_i$'s to be found with $\mathbf{b}_N$ known. In addition, the $\mathbf{a}_i$'s are related to the $\mathbf{b}_i$'s via

$$\mathbf{a}_i = \tilde{\overline{\mathbf{R}}}_{i,i-1} \cdot \overline{\beta}_{i,i+1} \cdot \mathbf{b}_i. \tag{8.7.30}$$

In this manner, all the fields in every region can be found. Note that the $\overline{\mathbf{M}}$ matrices defined above account for multiple reflections in the layered medium (see Exercise 8.31).

Hence, if a source is embedded in one of the layers, the combination of solutions from Figure 8.7.4 and Figure 8.7.5 can be used to calculate the field everywhere. Also, the above algorithm can be easily adapted to vector electromagnetic fields.

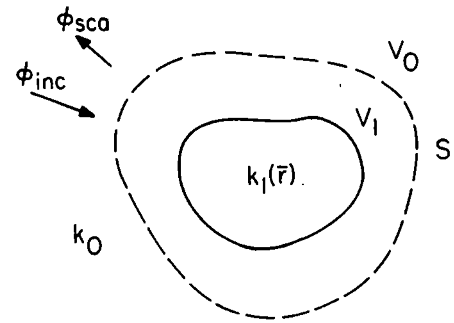


Figure 8.8.1 The scattering from an inhomogeneous scatterer is solvable by the combination of the surface integral equation method and the finite-element method.

§8.8 Surface Integral Equation with Finite-Element Method

The finite-element method (FEM) is versatile for highly inhomogeneous media. Moreover, it generates a sparse matrix for a differential equation economizing on memory requirement. But in using the FEM directly for an infinite domain, a large number of unknowns is usually involved, hence making the memory requirement inordinately large. Fortunately, one way to reduce the size of the problem is to couple the FEM with the surface integral equation method. In this manner, the FEM needs to be applied only over a finite domain drastically reducing the required memory. Such a method is also called the ***hybrid method*** (Silvester and Hsieh 1971; McDonald and Wexler 1972) or the ***unimoment method*** (Mei 1974; Chang and Mei 1976; Morgan and Mei 1979).

Consider the scattering by an inhomogeneous scatterer as shown in Figure 8.8.1. In the region exterior to S, we define a Green's function satisfying

$$(\nabla^2 + k_0^2)\, g_0(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') \tag{8.8.1}$$

and the radiation condition at infinity, and the field satisfying

$$(\nabla^2 + k_0^2)\, \phi_0(\mathbf{r}) = 0. \tag{8.8.2}$$

But interior to S, we define a Green's function satisfying

$$[\nabla \cdot p(\mathbf{r})\nabla + k_1^2(\mathbf{r})]g_1(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') \tag{8.8.3}$$

and the field satisfying

$$[\nabla \cdot p(\mathbf{r})\nabla + k_1^2(\mathbf{r})]\phi_1(\mathbf{r}) = 0. \tag{8.8.4}$$

Then, it is straightforward to show, as in (8.1.10) and (8.1.11), that

$$\left.\begin{array}{l}\mathbf{r} \in V_0,\ \phi_0(\mathbf{r}) \\ \mathbf{r} \in V_1,\ 0\end{array}\right\} = \phi_{inc}(\mathbf{r}) - \int_S dS' \left[g_0(\mathbf{r},\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_0(\mathbf{r}') - \phi_0(\mathbf{r}')\,\hat{n}'\cdot\nabla' g_0(\mathbf{r},\mathbf{r}')\right], \tag{8.8.5}$$

and that

$$\left.\begin{array}{l}\mathbf{r} \in V_1,\ \phi_1(\mathbf{r}) \\ \mathbf{r} \in V_0,\ 0\end{array}\right\} = \int_S dS' \left[g_1(\mathbf{r},\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}')\,\hat{n}'\cdot\nabla' g_1(\mathbf{r},\mathbf{r}')\right]. \tag{8.8.6}$$

In the above, we have assumed that $p(\mathbf{r}) = 1$ on S. In other words, $p(\mathbf{r}) \neq 1$ only in the scatterer depicted in Figure 8.8.1. Consequently, the above surface integral equations divide the problem into two problems, one internal to S and one external to S. In theory, it could be solved as before. But inside S, the medium is inhomogeneous and $g_1(\mathbf{r},\mathbf{r}')$ is not generally available in closed form.

Therefore, in order to find $g_1(\mathbf{r},\mathbf{r}')$, one presumably can solve (3) with a numerical method like the finite-element or the Galerkin's method. But to derive (6), $g_1(\mathbf{r},\mathbf{r}')$ has to satisfy (3) only for $\mathbf{r}$ and $\mathbf{r}'$ inside S. Hence, Equation (3) needs to be solved only over a finite region. For instance, it could be solved with the imposition of the natural boundary condition $\hat{n}\cdot\nabla' g_1(\mathbf{r},\mathbf{r}') = 0$ on S as discussed in Chapter 5. In this case, we let

$$g_1(\mathbf{r},\mathbf{r}') = \sum_{n=1}^{N} a_n f_n(\mathbf{r}), \quad \mathbf{r},\mathbf{r}' \in V_1, \tag{8.8.7}$$

where $f_n(\mathbf{r})$ constitutes a basis set that can approximate $g_1(\mathbf{r},\mathbf{r}')$ fairly well. Consequently, the matrix equation corresponding to (3) with the aforementioned natural boundary condition is (see Exercise 8.32)

$$\sum_{n=1}^{N} L_{mn} a_n = b_m, \quad m = 1,\ldots,N, \tag{8.8.8}$$

where

$$L_{mn} = -\langle \nabla f_m(\mathbf{r}), p(\mathbf{r})\nabla f_n(\mathbf{r})\rangle + \langle f_m(\mathbf{r}), k_1^2(\mathbf{r}) f_n(\mathbf{r})\rangle \tag{8.8.9a}$$

is symmetric, and

$$b_m = -\langle f_m(\mathbf{r}), \delta(\mathbf{r}-\mathbf{r}')\rangle = -f_m(\mathbf{r}'). \tag{8.8.9b}$$

As a note, the inner products above are defined as volume integrals in the volume bounded by S, i.e., $\langle f(\mathbf{r}), g(\mathbf{r})\rangle = \int_{V_1} d\mathbf{r}\, f(\mathbf{r})\, g(\mathbf{r})$.

Finally, we obtain

$$g_1(\mathbf{r},\mathbf{r}') = -\mathbf{f}^t(\mathbf{r})\cdot\overline{\mathbf{L}}^{-1}\cdot\mathbf{f}(\mathbf{r}'), \tag{8.8.10}$$

where $\overline{\mathbf{L}}$ is a matrix with elements L_{mn} and $\mathbf{f}$ is a column vector containing f_m.

Now, given $g_1(\mathbf{r},\mathbf{r}')$ and the natural boundary condition, $\hat{n}\cdot\nabla' g_1(\mathbf{r},\mathbf{r}') = 0$, it satisfies, the upper part of Equation (6) becomes

$$\phi_1(\mathbf{r}) = -\mathbf{f}^t(\mathbf{r})\cdot\overline{\mathbf{L}}^{-1}\cdot\int_S dS'\,\mathbf{f}(\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_1(\mathbf{r}'). \tag{8.8.11}$$

Note that the second term on the right-hand side of (6) vanishes by virtue of $\hat{n}\cdot\nabla' g_1(\mathbf{r},\mathbf{r}') = 0$ on S. The above and (5) together constitute the integral equations which can be solved for $\phi_1(\mathbf{r})$ and $\hat{n}\cdot\nabla\phi_1(\mathbf{r})$.

To solve (5) and (11), we expand the surface unknowns $\phi_1(\mathbf{r})$ and $\hat{n}\cdot\nabla\phi_1(\mathbf{r})$ as[14]

$$\phi_1(\mathbf{r}) = \sum_{m=1}^{M} c_m \psi_m(\mathbf{r}), \tag{8.8.12a}$$

$$\hat{n}\cdot\nabla\phi_1(\mathbf{r}) = \sum_{m=1}^{M} d_m \psi_m(\mathbf{r}), \tag{8.8.12b}$$

where $\psi_m(\mathbf{r})$ constitutes a basis set that can approximate $\phi_1(\mathbf{r})$ and $\hat{n}\cdot\nabla\phi_1(\mathbf{r})$ on S fairly well. The above can be substituted into (11) and tested with $\psi_n(\mathbf{r})$ on S, thereby yielding

$$\sum_{m=1}^{M}\langle\psi_n(\mathbf{r}),\psi_m(\mathbf{r})\rangle c_m = -\left\langle\psi_n(\mathbf{r}),\mathbf{f}^t(\mathbf{r})\right\rangle\cdot\overline{\mathbf{L}}^{-1}\cdot\sum_{m=1}^{M} d_m\left\langle\mathbf{f}(\mathbf{r}'),\psi_m(\mathbf{r}')\right\rangle, \tag{8.8.13}$$

where the inner product involves a surface integral over S. Furthermore, the above could be written as

$$\left\langle\boldsymbol{\psi}(\mathbf{r}),\boldsymbol{\psi}^t(\mathbf{r})\right\rangle\cdot\mathbf{c} = -\left\langle\boldsymbol{\psi}(\mathbf{r}),\mathbf{f}^t(\mathbf{r})\right\rangle\cdot\overline{\mathbf{L}}^{-1}\cdot\left\langle\mathbf{f}(\mathbf{r}'),\boldsymbol{\psi}^t(\mathbf{r}')\right\rangle\cdot\mathbf{d}, \tag{8.8.14}$$

where $\boldsymbol{\psi}(\mathbf{r})$, $\mathbf{c}$, and $\mathbf{d}$ are column vectors containing $\psi_m(\mathbf{r})$, c_m, and d_m respectively. Alternatively, (14) is equivalent to

$$\overline{\mathbf{F}}\cdot\mathbf{c} = -\overline{\mathbf{A}}\cdot\overline{\mathbf{L}}^{-1}\cdot\overline{\mathbf{A}}^t\cdot\mathbf{d} = -\overline{\mathbf{M}}\cdot\mathbf{d}, \tag{8.8.15}$$

where

$$\overline{\mathbf{F}} = \langle\boldsymbol{\psi}(\mathbf{r}),\boldsymbol{\psi}^t(\mathbf{r})\rangle, \tag{8.8.16a}$$

[14] It is not necessary to use the same basis set, $\psi_m(\mathbf{r})$, to expand both surface unknowns as we have done here.

$$\overline{\mathbf{A}} = \langle \psi(\mathbf{r}), \mathbf{f}^t(\mathbf{r}) \rangle, \tag{8.8.16b}$$

$$\overline{\mathbf{M}} = \overline{\mathbf{A}} \cdot \overline{\mathbf{L}}^{-1} \cdot \overline{\mathbf{A}}^t, \tag{8.8.16c}$$

and $\overline{\mathbf{F}}$ is an $M \times M$ matrix while $\overline{\mathbf{A}}$ is an $M \times N$ matrix where M is usually less than N.

Next, using the continuity of the potential plus the continuity of the normal derivative of the potential, we can express

$$\phi_0(\mathbf{r}) = \sum_{m=1}^{M} c_m \psi_m(\mathbf{r}), \tag{8.8.17a}$$

$$\hat{n} \cdot \nabla\phi_0(\mathbf{r}) = \sum_{m=1}^{M} d_m \psi_m(\mathbf{r}). \tag{8.8.17b}$$

Then, using the above in the lower half of Equation (5) and testing with $\psi_n(\mathbf{r})$ on S, we have

$$\begin{aligned}\langle \psi_n(\mathbf{r}), \phi_{inc}(\mathbf{r}) \rangle &= \sum_{m=1}^{M} \langle \psi_n(\mathbf{r}), g_0(\mathbf{r},\mathbf{r}'), \psi_m(\mathbf{r}) \rangle d_m \\ &- \sum_{m=1}^{M} \langle \psi_n(\mathbf{r}), \hat{n}' \cdot \nabla' g_0(\mathbf{r},\mathbf{r}'), \psi_m(\mathbf{r}) \rangle c_m, \quad n = 1, \ldots, M. \end{aligned} \tag{8.8.18}$$

Note that the above is just a matrix equation of the form

$$\boldsymbol{\phi}_{inc} = \overline{\mathbf{g}} \cdot \mathbf{d} - \overline{\mathbf{N}} \cdot \mathbf{c}, \tag{8.8.19}$$

where

$$[\boldsymbol{\phi}_{inc}]_n = \langle \psi_n(\mathbf{r}), \phi_{inc}(\mathbf{r}) \rangle, \tag{8.8.20a}$$

$$[\overline{\mathbf{g}}]_{nm} = \langle \psi_n(\mathbf{r}), g_0(\mathbf{r},\mathbf{r}'), \psi_m(\mathbf{r}') \rangle, \tag{8.8.20b}$$

$$[\overline{\mathbf{N}}]_{nm} = \langle \psi_n(\mathbf{r}), \hat{n}' \cdot \nabla' g_0(\mathbf{r},\mathbf{r}'), \psi_m(\mathbf{r}') \rangle. \tag{8.8.20c}$$

Moreover, in the above, we have defined

$$\langle \psi_n(\mathbf{r}), f(\mathbf{r},\mathbf{r}'), \psi_m(\mathbf{r}') \rangle = \int_S dS \int_S dS' \, \psi_n(\mathbf{r}) f(\mathbf{r},\mathbf{r}') \psi_m(\mathbf{r}'). \tag{8.8.20d}$$

Now, (11) and (5) have been reduced to matrix equations (15) and (19) respectively, from which $\mathbf{c}$ and $\mathbf{d}$ can be solved. Therefore, on eliminating $\mathbf{d}$ between (15) and (19), we have

$$\boldsymbol{\phi}_{inc} = -\left[\overline{\mathbf{g}} \cdot \overline{\mathbf{M}}^{-1} \cdot \overline{\mathbf{F}} + \overline{\mathbf{N}}\right] \cdot \mathbf{c}, \tag{8.8.21}$$

or

$$\mathbf{c} = -\left[\overline{\mathbf{g}} \cdot \overline{\mathbf{M}}^{-1} \cdot \overline{\mathbf{F}} + \overline{\mathbf{N}}\right]^{-1} \cdot \boldsymbol{\phi}_{inc}, \tag{8.8.22a}$$

$$\mathbf{d} = -\overline{\mathbf{M}}^{-1} \cdot \overline{\mathbf{F}} \cdot \mathbf{c}. \tag{8.8.22b}$$

Once $\mathbf{c}$ and $\mathbf{d}$ are found, the surface unknowns can be found through (17) and the field everywhere determined via (5) and (11). This idea is, of course, easily extended to solving vector electromagnetic integral equations, albeit with increased complexity.

Notice that when $g_1(\mathbf{r},\mathbf{r}')$ was solved for in (3), it was done so with the natural boundary condition $\hat{n} \cdot \nabla g_1(\mathbf{r},\mathbf{r}') = 0$. This is actually equivalent to a source excitation problem in a cavity with impenetrable walls. Unfortunately, this cavity has resonant frequencies that are purely real in the lossless case. Therefore, at the resonant frequencies of the cavity formed by S, the matrix $\overline{\mathbf{L}}$ becomes singular. When this happens, it is quite difficult to solve for $\overline{\mathbf{M}}$ in (15).

When $\mathbf{c}$ and $\mathbf{d}$ are found in (22), only $\overline{\mathbf{M}}^{-1}$ is required. In this case, the singular value decomposition method may be used to find the inverse of $\overline{\mathbf{L}}$ (see Exercise 8.33). Moreover, the regularization method may be used to find the inverse of $\overline{\mathbf{L}}$ (Exercise 8.34; also see Tikhonov 1963). Alternatively, an impedance boundary condition for the Green's function, instead of the natural boundary condition, will eliminate the resonance problem (see Exercise 8.35), because then the cavity is lossy with a complex resonant frequency.

We have illustrated how the surface-integral-equation method is used to merge a finite-element solution with the solution in the external region. But the surface integral equations yield matrices $\overline{\mathbf{g}}$ and $\overline{\mathbf{N}}$ which are dense, in contrast to the sparse matrix $\overline{\mathbf{L}}$ in (9) generated by the finite-element method. Hence, this precludes the use of a sparse-matrix solver which is usually more efficient than a dense-matrix solver. To remedy this, absorbing boundary conditions defined in Chapter 4 may alternatively be used to merge the finite-element solution with the exterior solution (see Exercise 8.36). This then yields sparse matrices which can be inverted with sparse-matrix solvers.

§8.9 Volume Integral Equations

When a bounded medium is highly inhomogeneous, there are several methods of solving for its scattering solution. One way is to approximate the inhomogeneous medium with N scatterers and seek its scattering solution via the method of Section 8.6. If the inhomogeneous body can be approximated by a multilayered medium, the method expounded in Section 8.7 can be used. Furthermore, the hybrid method of Section 8.8 may be used. An alternative approach is to use volume integral equations where the unknowns in the problem are expressed in terms of volume current flowing in the inhomogeneity. The volume current consists of conduction current as well as displacement current induced by the total electric field. An integral equation can then be formulated from which the total field is solved. We shall first show how such an integral equation can be formulated for the scalar wave equation and later, formulate the integral equation for the electromagnetic wave case. Historically, the volume integral equation method has been developed as early as 1913 by Esmarch (see Born and Wolf 1980, p. 98). This equation is also described by Richmond (1965a, b), Harrington (1968), Poggio and Miller (1973), and Ström (1975).

The volume integral equation offers an alternative physical picture of the mechanism that gives rise to scattering. As such, it provides insight as to how approximate scattering solutions can be obtained, as shall be illustrated in the next section. Furthermore, it can be used to formulate inverse scattering algorithms detailed in the next chapter.

§§8.9.1 *Scalar Wave Case*

We shall first derive the volume integral equation for the scalar wave case. In this case, the pertinent scalar wave equation is

$$[\nabla^2 + k^2(\mathbf{r})]\phi(\mathbf{r}) = q(\mathbf{r}), \tag{8.9.1}$$

where $k^2(\mathbf{r}) = \omega^2\mu(\mathbf{r})\epsilon(\mathbf{r})$ represents an inhomogeneous medium over a finite domain V, and $k^2 = k_b^2 = \omega^2\mu_b\epsilon_b$ outside V (see Figure 8.9.1). Next, we define a Green's function satisfying

$$[\nabla^2 + k_b^2]g(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}'). \tag{8.9.2}$$

Then, Equation (1) can be rewritten as

$$[\nabla^2 + k_b^2]\phi(\mathbf{r}) = q(\mathbf{r}) - [k^2(\mathbf{r}) - k_b^2]\phi(\mathbf{r}). \tag{8.9.3}$$

Note that the right-hand side of (3) can be considered an equivalent source. Since the Green's function corresponding to the differential operator on the left-hand side of (3) is known, by the principle of linear superposition, we can write

$$\phi(\mathbf{r}) = -\int_{V_s} dV' g(\mathbf{r},\mathbf{r}')q(\mathbf{r}') + \int_V dV' g(\mathbf{r},\mathbf{r}')[k^2(\mathbf{r}') - k_b^2]\phi(\mathbf{r}'). \tag{8.9.4}$$

The first term on the right-hand side is just the field due to the source in the absence of the inhomogeneity, and hence, is the incident field. Therefore, Equation (4) becomes

$$\phi(\mathbf{r}) = \phi_{inc}(\mathbf{r}) + \int_V dV' g(\mathbf{r},\mathbf{r}')[k^2(\mathbf{r}') - k_b^2]\phi(\mathbf{r}'). \tag{8.9.5}$$

In the above equation, if the total field $\phi(\mathbf{r}')$ inside the volume V is known, then $\phi(\mathbf{r})$ can be calculated everywhere. But $\phi(\mathbf{r})$ is unknown at this point.

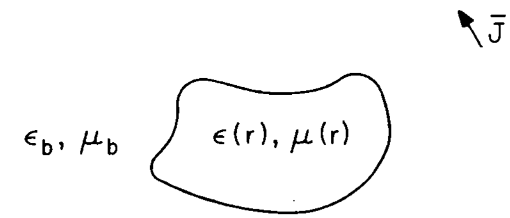


Figure 8.9.1 A current source radiating in the vicinity of a general inhomogeneity.

To solve for $\phi(\mathbf{r})$, an integral equation has to be formulated for $\phi(\mathbf{r})$. To this end, we imposed (5) for $\mathbf{r}$ in V. Then, $\phi(\mathbf{r})$ on the left-hand side and on the right-hand side are the same unknown defined over the same domain. Consequently, Equation (5) becomes the desired integral equation

$$\phi_{inc}(\mathbf{r}) = \phi(\mathbf{r}) - \int_V dV' g(\mathbf{r},\mathbf{r}')[k^2(\mathbf{r}') - k_b^2]\phi(\mathbf{r}'), \quad \mathbf{r} \in V. \tag{8.9.6}$$

In the above, the unknown $\phi(\mathbf{r})$ is defined over a volume V, over which the integration is performed, and hence the name, volume integral equation. Alternatively, the above can be rewritten as

$$\phi_{inc}(\mathbf{r}) = [\mathcal{I} - \mathcal{L}(\mathbf{r},\mathbf{r}')]\phi(\mathbf{r}'), \quad \mathbf{r} \in V, \tag{8.9.7}$$

where $\mathcal{I}$ is an identity operator while $\mathcal{L}$ is the integral operator in (6). It is also a ***Fredholm integral equation*** of the second kind because the unknown is both inside and outside the integral operator.

§§8.9.2 *The Electromagnetic Wave Case*

We shall show how the corresponding integral equation can be derived for a finite size, inhomogeneous scatterer for the electromagnetic wave case shown in Figure 8.9.1. First, from Maxwell's equations, it follows that the electric field everywhere satisfies the following equation:

$$\nabla \times \mu^{-1}\nabla \times \mathbf{E}(\mathbf{r}) - \omega^2\epsilon\mathbf{E}(\mathbf{r}) = i\omega\mathbf{J}(\mathbf{r}), \tag{8.9.8}$$

where μ and ϵ are functions of position inside the inhomogeneous region V. Next, subtracting $\nabla\times\mu_b^{-1}\nabla\times\mathbf{E}(\mathbf{r}) - \omega^2\epsilon_b\mathbf{E}(\mathbf{r})$ from both sides of the equation, we have

$$\nabla\times(\mu^{-1}-\mu_b^{-1})\nabla\times\mathbf{E}(\mathbf{r}) - \omega^2(\epsilon-\epsilon_b)\mathbf{E}(\mathbf{r}) = i\omega\mathbf{J}(\mathbf{r}) - \nabla\times\mu_b^{-1}\nabla\times\mathbf{E}(\mathbf{r}) + \omega^2\epsilon_b\mathbf{E}(\mathbf{r}), \tag{8.9.9}$$

To formulate the integral equation, we need the dyadic Green's function to the problem in the absence of the scatterer. The dyadic Green's function satisfies the equation

$$\nabla \times \mu_b^{-1}\nabla \times \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') - \omega^2\epsilon_b\,\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \mu_b^{-1}\overline{\mathbf{I}}\,\delta(\mathbf{r}-\mathbf{r}'). \tag{8.9.10}$$

Even though μ_b and ϵ_b need not be constant, but when they are constant, the solution to (10) is well known, as discussed in Chapters 1 and 7. Consequently, we can rewrite Equation (9) as

$$\nabla\times\mu_b^{-1}\nabla\times\mathbf{E}(\mathbf{r})-\omega^2\epsilon_b\mathbf{E}(\mathbf{r}) = i\omega\mathbf{J}(\mathbf{r})+\omega^2(\epsilon-\epsilon_b)\mathbf{E}(\mathbf{r})-\nabla\times\left(\frac{1}{\mu}-\frac{1}{\mu_b}\right)\nabla\times\mathbf{E}(\mathbf{r}). \tag{8.9.11}$$

Physically, the terms on the right-hand side of Equation (11) are effective current sources. Therefore, analogous to (7.4.5a), the solution to (11) is

$$\mathbf{E}(\mathbf{r}) = i\omega\int_V d\mathbf{r}'\,\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\mu_b\mathbf{J}(\mathbf{r}') + \omega^2\int_V d\mathbf{r}'\,\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\mu_b(\epsilon-\epsilon_b)\mathbf{E}(\mathbf{r}')$$
$$-\int_V d\mathbf{r}'\,\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\mu_b\nabla'\times\left(\frac{1}{\mu}-\frac{1}{\mu_b}\right)\nabla'\times\mathbf{E}(\mathbf{r}'). \tag{8.9.12}$$

In the above, the first term is just the incident field; hence, (12) becomes

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{inc}(\mathbf{r}) + \omega^2\int_V d\mathbf{r}'\,\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\mu_b(\epsilon-\epsilon_b)\mathbf{E}(\mathbf{r}')$$
$$-\int_V d\mathbf{r}'\,\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\mu_b\nabla'\times\left(\frac{1}{\mu}-\frac{1}{\mu_b}\right)\nabla'\times\mathbf{E}(\mathbf{r}'). \tag{8.9.13}$$

The integrals in (13) are contributions to the field $\mathbf{E}$ from the volume current induced in the scatterer by the total electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ (note that $\nabla\times\mathbf{E} = i\omega\mu\mathbf{H}$). Hence, the first term is generated by the electric polarization current or displacement current, while the second term is generated by the magnetic polarization charges (see Exercise 8.37). Moreover, when the scatterer is conductive such that $\epsilon = \epsilon' + i\sigma/\omega$, the first integral in (13) is due to the conduction current induced by the field as well. This is obviated by substituting the complex permittivity into (13) and identifying a term proportional to $\sigma\mathbf{E}$ corresponding to conduction currents. On the other hand, if $\mu = \mu_b$, Equation (13) simplifies to

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{inc}(\mathbf{r}) + \int_V d\mathbf{r}'\,\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot O(\mathbf{r}')\mathbf{E}(\mathbf{r}'), \tag{8.9.14}$$

where $O(\mathbf{r}') = \omega^2(\mu\epsilon - \mu_b\epsilon_b) = k^2(\mathbf{r}') - k_b^2$.

In Equations (13) and (14), the field $\mathbf{E}_{inc}$ is usually known since we know the source $\mathbf{J}$. But the total field $\mathbf{E}(\mathbf{r})$ is unknown, and it is in the integral as well. Therefore, analogous to (6), (14) is a volume integral equation, which can be written as

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{inc}(\mathbf{r}) - \overline{\mathcal{L}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{E}(\mathbf{r}'), \qquad \mathbf{r}'\in V, \quad \mathbf{r}\in V, \tag{8.9.15}$$

where $\overline{\mathcal{L}}$ is a linear integral operator in (13) or (14). Alternatively, we can write (15) as

$$\mathbf{E}_{inc}(\mathbf{r}) = \left[\overline{\mathcal{I}} - \overline{\mathcal{L}}(\mathbf{r},\mathbf{r}')\right]\cdot\mathbf{E}(\mathbf{r}'), \qquad \mathbf{r}'\in V, \quad \mathbf{r}\in V, \tag{8.9.16}$$

where $\overline{\mathcal{I}}$ is an identity operator. Since $\overline{\mathcal{I}}-\overline{\mathcal{L}}$ is a linear operator, we can apply Galerkin's method, or the method of moments to solve (16), as discussed in Chapter 5. Once $\mathbf{E}(\mathbf{r})$ is known inside V, $\mathbf{E}$ can be found everywhere via Equations (13) or (14). Equation (16) is a Fredholm integral equation of the second kind because the unknown is both inside and outside the integral. Equation (14) can also be written in operator form as shown in Subsection 9.3.3 of Chapter 9.

The above derivation is easily generalized to the case where the dyadic Green's function is for layered media discussed in Chapter 7. In this case, the background medium need not be homogeneous.

§§8.9.3 *Matrix Representation of the Integral Equation*

Given the integral equation in (14), it can be converted into a matrix equation quite easily using the method discussed in Chapter 5, i.e., by projecting the integral operator onto a space spanned by $\mathbf{E}_n(\mathbf{r})$, where $\mathbf{E}_n(\mathbf{r}) = 0$ for $\mathbf{r}\notin V$. To this end, we let

$$\mathbf{E}(\mathbf{r}) = \sum_n a_n\mathbf{E}_n(\mathbf{r}), \qquad \mathbf{r}\in V, \tag{8.9.17}$$

in (14). Then,

$$\sum_n a_n\mathbf{E}_n(\mathbf{r}) = \mathbf{E}_{inc}(\mathbf{r}) + \sum_n a_n\int_V d\mathbf{r}'\,\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot O(\mathbf{r}')\mathbf{E}_n(\mathbf{r}'). \tag{8.9.18}$$

The above integral operator acting on $\mathbf{E}_n(\mathbf{r})$ is not a symmetric operator. Nevertheless, it can by symmetrized by multiplying (18) by $O(\mathbf{r})$ (see Exercise 8.38). In this manner, (18) becomes

$$\sum_n a_nO(\mathbf{r})\mathbf{E}_n(\mathbf{r}) = O(\mathbf{r})\mathbf{E}_{inc}(\mathbf{r})$$
$$+\sum_n a_nO(\mathbf{r})\int_V d\mathbf{r}'\,\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot O(\mathbf{r}')\mathbf{E}_n(\mathbf{r}'). \tag{8.9.19}$$

Consequently, after dot-multiplying the above by $\mathbf{E}_m(\mathbf{r})$ and integrating as in Galerkin's method, we have

$$\sum_n a_n\langle\mathbf{E}_m, O(\mathbf{r})\mathbf{E}_n\rangle = \langle\mathbf{E}_m, O(\mathbf{r})\mathbf{E}_{in}\rangle$$
$$+\sum_n a_n\left\langle\mathbf{E}_m, O(\mathbf{r})\int_V d\mathbf{r}'\,\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot O(\mathbf{r}')\mathbf{E}_n(\mathbf{r}')\right\rangle. \tag{8.9.20}$$

Now, Equation (20) is a matrix equation of the form

$$\sum_n M_{mn} a_n = b_m + \sum_n N_{mn} a_n, \tag{8.9.21}$$

or

$$(\overline{\mathbf{M}} - \overline{\mathbf{N}}) \cdot \mathbf{a} = \mathbf{b}, \tag{8.9.21a}$$

where

$$M_{mn} = \langle \mathbf{E}_m, O(\mathbf{r})\mathbf{E}_n \rangle, \tag{8.9.22a}$$

$$N_{mn} = \left\langle \mathbf{E}_m, O(\mathbf{r}) \int_V d\mathbf{r}'\, \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \cdot O(\mathbf{r}')\mathbf{E}_n(\mathbf{r}') \right\rangle, \tag{8.9.22b}$$

$$b_m = \langle \mathbf{E}_m, O(\mathbf{r})\mathbf{E}_{inc} \rangle. \tag{8.9.22c}$$

Moreover, with a finite basis set, we can always invert (21a) to find the unknown $\mathbf{a}$, the column vector that contains the unknowns a_n's.

Because of the singularity of the dyadic Green's function $\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')$, (22b) is not well defined (see Chapter 7). This can be remedied, however, by writing the integral as

$$N_{mn} = \int_V d\mathbf{r}\, O(\mathbf{r})\mathbf{E}_m(\mathbf{r}) \cdot \left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k_b^2}\right) \cdot \int_V d\mathbf{r}' g(\mathbf{r},\mathbf{r}') O(\mathbf{r}')\mathbf{E}_n(\mathbf{r}'). \tag{8.9.23}$$

Furthermore, using integration by parts on the term that contains $\nabla\nabla$, we have (see Exercise 8.38)

$$\begin{aligned} N_{mn} = & \int_V d\mathbf{r}\, O(\mathbf{r})\mathbf{E}_m(\mathbf{r}) \cdot \int_V d\mathbf{r}\, g(\mathbf{r},\mathbf{r}')O(\mathbf{r}')\mathbf{E}_n(\mathbf{r}') \\ & - \frac{1}{k_b^2} \int_V d\mathbf{r}\, \nabla \cdot [O(\mathbf{r})\mathbf{E}_m(\mathbf{r})] \int_V d\mathbf{r}' g(\mathbf{r},\mathbf{r}') \nabla' \cdot [O(\mathbf{r}')\mathbf{E}_n(\mathbf{r}')]. \end{aligned} \tag{8.9.24}$$

This is done to circumvent having to integrate the singularity of the dyadic Green's function. Note further that in (21a), $\overline{\mathbf{M}}$ and $\overline{\mathbf{N}}$ matrices are symmetrical. A real-symmetric matrix readily lends itself to being solved by the conjugate gradient method.

Equation (19) is also solvable by testing it with other weighting functions. A popular testing function is the Dirac delta function, as in the method of point matching. In this case, the double integrals in (24) reduce to single integrals, and the effort to compute the matrix element in the matrix equation is greatly reduced.

§8.10 Approximate Solutions of the Scattering Problem

The solution of the volume integral equation usually has to be solved for numerically. This is, in general, computationally intensive because in

finding the matrix element N_{mn} in the previous section, we may have to perform a double integration. For many problems, however, especially when the scattering from the inhomogeneity is weak, it suffices to derive approximate solutions to the scattering problem. Therefore, we shall discuss two approximate solutions, the ***Born approximation*** (Born and Wolf 1980, p. 453), which works better at low frequencies, and the ***Rytov approximation*** (Tatarski 1961), which works better at higher frequencies. Moreover, both approximations are weak scatterer approximations.

§§8.10.1 Born Approximation

In the cases when $k^2 - k_b^2$ is small, or where the contrast of the scatterer is weak so that the second term on the right of Equation (8.9.14) is small compared to the first term, we can approximate

$$\mathbf{E}(\mathbf{r}) \simeq \mathbf{E}_{inc}(\mathbf{r}). \tag{8.10.1}$$

Then, the total field in Equation (8.9.14) can be approximately calculated as

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_{inc}(\mathbf{r}) + \int_V d\mathbf{r}'\, \overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \cdot O(\mathbf{r}')\mathbf{E}_{inc}(\mathbf{r}'). \tag{8.10.2}$$

The above is known as the first-order Born approximation. It is also the first-order approximation in the Neumann series expansion of integral equation in (8.9.14), or the Taylor series approximation of $\mathbf{E}(\mathbf{r})$ using $(k^2 - k_b^2)$ as a small parameter (see Exercise 8.39; also see Chapter 9, Subsection 9.3.3).

Since the Born approximation is good only when the second term is much smaller than the first term in (2), we can establish the regime of validity of the Born approximation.[15] First, notice that for the homogeneous background case,

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k_b^2}\right) g(\mathbf{r},\mathbf{r}'). \tag{8.10.3}$$

If the size of the scatterer is of the order L, and $k_b L \ll 1$, then by dimensional analysis (see Exercise 8.40),

$$g(\mathbf{r},\mathbf{r}') \sim \frac{1}{L}, \quad \nabla\nabla \sim \frac{1}{L^2}. \tag{8.10.4}$$

Then,

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \sim \left(1 + \frac{1}{k_b^2 L^2}\right) \frac{1}{L}, \tag{8.10.5a}$$

$$O(\mathbf{r}) = (k^2 - k_b^2) \sim k_b^2 \Delta\epsilon_r, \tag{8.10.5b}$$

[15] The regime of validity of the Born approximation has also been discussed by Keller (1969).

(where $\Delta\epsilon_r = \epsilon/\epsilon_b - 1$) and

$$\int d\mathbf{r}' \sim L^3. \tag{8.10.5c}$$

Therefore, one sees that the second term in (2) is of the order of

$$[(k_b L)^2 + 1]\Delta\epsilon_r E_{inc}. \tag{8.10.6}$$

But since $k_b L \ll 1$, in order for the scattered field to be much smaller than the incident field, the constraint is

$$\Delta\epsilon_r \ll 1, \tag{8.10.7}$$

in the long wavelength limit.

On the other hand, in the short wavelength limit, the controlling factor for the magnitude of $\nabla\nabla$ is not the size of the object, but the field variation inside the object, which is controlled by the wavelength of the field. Then,

$$\nabla\nabla \sim k_b^2. \tag{8.10.8}$$

Furthermore, the phase of the wave as it propagates inside the object becomes important. For example, if $\mathbf{E}_{inc} \sim e^{i\mathbf{k}_b\cdot\mathbf{r}}$, then the total field inside a tenuous inhomogeneity consists of a linear superposition of plane waves of the type $\mathbf{E} \sim e^{i\mathbf{k}\cdot\mathbf{r}}$ (which is motivated by WKB type approximation; see Chapter 2) when $k \to \infty$.[16] Therefore, we can write

$$\begin{aligned} \mathbf{E} &\sim e^{i\mathbf{k}_b\cdot\mathbf{r}} e^{i(\mathbf{k}-\mathbf{k}_b)\cdot\mathbf{r}} \\ &\sim \mathbf{E}_{inc} e^{i(\mathbf{k}-\mathbf{k}_b)\cdot\mathbf{r}}, \end{aligned} \tag{8.10.9}$$

and hence, $\mathbf{E} \simeq \mathbf{E}_{inc}$ only if $(k - k_b)L \ll 1$. Consequently, at high frequencies, the Born approximation is valid only if

$$k_b L \Delta\epsilon_r \ll 1, \quad k_b L \to \infty. \tag{8.10.10}$$

Note further that the above is a much more stringent restriction than (7) (also see Exercise 8.41).

In some applications where there is no charge accumulation (for example, a TM wave impinging on a cylinder), the $\nabla\nabla$ term can be neglected. Then, the problem reduces to a scalar one, and the constraint on the Born approximation from (4) to (6) becomes (Exercise 8.42)

$$k_b^2 L^2 \Delta\epsilon_r \ll 1. \tag{8.10.11}$$

Moreover, in the long wavelength limit, $k_b L \ll 1$, and this constraint could be met even when $\Delta\epsilon_r > 1$. Hence, in this case and the scalar wave case,

[16] Strictly speaking, it should be $\mathbf{E} \sim e^{i\int k dz'}$, but this distinction is not important for this order-of-magnitude argument.

the Born approximation becomes exceedingly good at low frequencies. But at high frequencies, constraint (10) holds true also for scalar waves, since polarization-charge effect is unimportant at high frequencies. Finally, it is important to note that the above constraints are for a three-dimensional space. In a one- or two-dimensional space, they have to be rederived (Exercise 8.42).

The Born approximation is a single-scattering approximation. Moreover, note that in Equation (2), the incident wave enters the scatterer with no distortion, induces the polarization current proportional to $(k^2 - k_b^2)\mathbf{E}_{inc}$, and causes a re-radiation or scattering. Since the incident field is unaffected while it gives rise to a scattered field, the Born approximation violates energy conservation. However, because of the symmetry of the dyadic Green's function, reciprocity is still preserved under the Born approximation.

When a conductive inhomogeneity is in an insulating background, $k^2(\mathbf{r}) \sim i\omega\mu\sigma(\mathbf{r})$ when $\omega \to 0$ and $k_b^2 = \omega^2\mu\epsilon_b$ when $\omega \to 0$. Then, from (3)

$$\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') \sim \frac{1}{\omega^2}, \qquad \omega \to 0. \tag{8.10.12}$$

However,

$$O(\mathbf{r}) \sim \omega, \qquad \omega \to 0. \tag{8.10.13}$$

Therefore, the scattered field term in (2) is proportional to $1/\omega$ when $\omega \to 0$. This low-frequency divergence implies that the Born approximation is exceedingly bad at low frequencies when a conductive inhomogeneity is embedded in an insulating background. This happens because when we approximate the induced conduction current (eddy current) in the conductive inhomogeneity by

$$\mathbf{J} = \sigma\mathbf{E} \simeq \sigma\mathbf{E}_{in}, \tag{8.10.14}$$

in the Born approximation, the induced eddy current is terminated abruptly at the insulator/conductor interface. We can see this from the continuity equation, where the charge $\varrho = \nabla\cdot\mathbf{J}/i\omega$ implying that these charges at the interface diverge as $1/\omega$ when $\omega \to 0$, giving rise to this low-frequency divergence if $\nabla\cdot\mathbf{J} \neq 0$. But note that this problem does not arise if both the background and the scatterer are conductive.

§§8.10.2 Rytov Approximation

We have seen in various instances (e.g., Section 2.8, Chapter 2) that the polarization-charge effect is unimportant at high frequencies when the wavelength is much smaller than the size of the inhomogeneity. If this is actually the case, the study of the vector electromagnetic wave equation may be reduced to the study of the scalar wave equation. Therefore, the pertinent equation is then

$$[\nabla^2 + k^2(\mathbf{r})]\phi(\mathbf{r}) = 0. \tag{8.10.15}$$

To derive the Rytov approximation, we first let

$$\phi(\mathbf{r}) = e^{i\psi(\mathbf{r})}. \tag{8.10.16}$$

Then,

$$\nabla\phi(\mathbf{r}) = i\phi(\mathbf{r})\nabla\psi(\mathbf{r}), \tag{8.10.17a}$$

$$\nabla\cdot\nabla\phi(\mathbf{r}) = \{i\nabla^2\psi(\mathbf{r}) - [\nabla\psi(\mathbf{r})]^2\}\phi(\mathbf{r}). \tag{8.10.17b}$$

Using the above in Equation (15), we have

$$i\nabla^2\psi(\mathbf{r}) - (\nabla\psi)^2 + k^2(\mathbf{r}) = 0. \tag{8.10.18}$$

At this point, Equation (18) is still exact but nonlinear. However, we can solve (18) perturbatively by letting

$$\psi(\mathbf{r}) \sim \psi_0(\mathbf{r}) + \psi_1(\mathbf{r}). \tag{8.10.19}$$

Here, $\psi_0(\mathbf{r})$ is assumed to satisfy the equation

$$i\nabla^2\psi_0(\mathbf{r}) - (\nabla\psi_0)^2 + k_b^2(\mathbf{r}) = 0, \tag{8.10.20}$$

i.e., it is the solution in some background medium with wave number k_b. After substituting (19) into (18), consequently, we have

$$i\nabla^2\psi_1(\mathbf{r}) - 2(\nabla\psi_0)\cdot(\nabla\psi_1) - (\nabla\psi_1)^2 + O(\mathbf{r}) = 0, \tag{8.10.21}$$

where $O(\mathbf{r}) = k^2 - k_b^2$. At this point, the above equation is nonlinear, but it could be simplified by using the identity that

$$\nabla^2(\phi_0\psi_1) = \psi_1\nabla^2\phi_0 + 2(\nabla\phi_0)\cdot(\nabla\psi_1) + \phi_0\nabla^2\psi_1, \tag{8.10.22}$$

where $\phi_0 = e^{i\psi_0(\mathbf{r})}$. Since $\nabla^2\phi_0 = -k_b^2\phi_0$ and $\nabla\phi_0 = i(\nabla\psi_0)\,\phi_0$, we have

$$\nabla^2(\phi_0\psi_1) = -k_b^2\psi_1\phi_0 + 2i\phi_0(\nabla\psi_0)\cdot(\nabla\psi_1) + \phi_0\nabla^2\psi_1. \tag{8.10.23}$$

Then, on multiplying (21) by $i\phi_0$ and using (23), we obtain

$$\nabla^2(\phi_0\psi_1) + k_b^2\phi_0\psi_1 = -i\phi_0(\nabla\psi_1)^2 + i\phi_0 O(\mathbf{r}). \tag{8.10.24}$$

Equation (24) is still exact at this point. But if we assume that ψ_1 is small so that $(\nabla\psi_1)^2$ is even smaller, Equation (24) can be approximated as

$$(\nabla^2 + k_b^2)\,\phi_0\psi_1 = i\phi_0 O(\mathbf{r}). \tag{8.10.25}$$

The solution to (25) is then

$$\psi_1(\mathbf{r}) = -\frac{i}{\phi_0(\mathbf{r})}\int d\mathbf{r}'\, g(\mathbf{r},\mathbf{r}')\,\phi_0(\mathbf{r}')O(\mathbf{r}'). \tag{8.10.26}$$

The above approximation is known as the Rytov approximation; the total solution is

$$\phi(\mathbf{r}) \simeq \phi_0(\mathbf{r})e^{i\psi_1(\mathbf{r})}. \tag{8.10.27}$$

The Rytov approximation is valid when the first term on the right of Equation (24) is much smaller than the second term, or

$$(\nabla\psi_1)^2 \ll O(\mathbf{r}). \tag{8.10.28}$$

Moreover, this approximation attempts to correct for the phase of the wave as it propagates through the inhomogeneous media. Hence, it shares some similarity with the WKB approximation.[17]

Applying dimensional analysis, it can be shown from (26) that

$$\psi_1(\mathbf{r}) \sim k_b^2L^2\Delta\epsilon_r, \quad \text{when} \quad k_bL \to 0. \tag{8.10.29}$$

Furthermore, assuming that $\nabla \sim 1/L$ when $k_bL \to 0$, then the use of (29) into (28) yields the condition that

$$(k_bL)^2\Delta\epsilon_r \ll 1, \tag{8.10.30}$$

which is the same as (11). Again, this is dependent on dimensions.

On the other hand, when the frequency tends to infinity, the field inside the inhomogeneity is of the form $e^{i\mathbf{k}\cdot\mathbf{r}}$. Then,

$$\phi(\mathbf{r}) \sim e^{i\mathbf{k}\cdot\mathbf{r}} \sim e^{i\mathbf{k}_b\cdot\mathbf{r}}e^{i(\mathbf{k}-\mathbf{k}_b)\cdot\mathbf{r}} \sim \phi_0 e^{i\psi_1(\mathbf{r})}. \tag{8.10.31}$$

Therefore, $\psi_1(\mathbf{r}) \simeq (\mathbf{k}-\mathbf{k}_b)\cdot\mathbf{r}$, and

$$\psi_1(\mathbf{r}) \sim k_bL\Delta\epsilon_r, \quad k_bL \to \infty. \tag{8.10.32}$$

Moreover, assuming that $\nabla \sim 1/L$ when $k_bL \to \infty$,[18] the use of (32) into (28) yields the constraint

$$\Delta\epsilon_r \ll 1, \tag{8.10.33}$$

which is more relaxed than (10).

In the Rytov approximation, the correction $\psi_1(\mathbf{r})$ occurs as a phase term. But the magnitude of the correction to $\phi_0(\mathbf{r})$ is always unity even when $\psi_1(\mathbf{r})$ is bad. Hence, the approximation breaks down more gracefully compared to the Born approximation. Furthermore, the form given by (27) or (31) is more suitable for the field inside the inhomogeneity. Outside the scatterer, the constraint is again given by (10) for high frequencies (Exercise 8.43).

Note that the Born approximation for the scalar wave equation is of the form

$$\phi_1(\mathbf{r}) = -\int d\mathbf{r}'\, g(\mathbf{r},\mathbf{r}')O(\mathbf{r}')\,\phi_0(\mathbf{r}'), \tag{8.10.34}$$

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[17] The regime of validity of the Rytov approximation has also been discussed by Fried (1967), Brown (1967), Keller (1969), and Crane (1976).

[18] Unlike (8), $\nabla \sim 1/L$ instead of k_b here because ∇ operates on ψ_1, which is the phase variation.

where $\phi_1(\mathbf{r})$ is the scattered field and $\phi_0(\mathbf{r})$ is the incident field. But when $\psi_1(\mathbf{r})$ is very small, we can rewrite (27) as

$$\phi(\mathbf{r}) \cong \phi_0(\mathbf{r}) + i\psi_1(\mathbf{r})\,\phi_0(\mathbf{r}). \tag{8.10.35}$$

Therefore, $\phi_1(\mathbf{r}) \sim i\psi_1(\mathbf{r})\,\phi_0(\mathbf{r})$ if $\psi_1(\mathbf{r})$ is very small. Then, on multiplying Equation (26) by $i\phi_0(\mathbf{r})$, we recover Equation (34). Hence, the Born and Rytov approximations reduce to the same approximation when the scattered field is very weak.

Both the Born and the Rytov approximations assume that the scattered field is linearly proportional to the inhomogeneity $O(\mathbf{r})$. As such, this linearized approximation makes them particularly suitable for solving the inverse problems when the scatterers are weak scatterers.

Exercises for Chapter 8

8.1 (a) Find another solution of (8.1.3) in V_1 that also satisfies the radiation condition at infinity.

(b) Show that the integral over S_{inf} in (8.1.6) vanishes by virtue of the radiation condition. In other words, if the sources that generate the field are finite in extent, all fields will look like outgoing plane waves when $\mathbf{r} \to \infty$.

8.2 (a) Derive Equation (8.1.11) in the manner of Equation (8.1.10). Does $g_2(\mathbf{r}-\mathbf{r}')$ in (8.1.11) need to satisfy the radiation condition?

(b) Show that (8.1.11) can be simplified by imposing either a homogeneous Dirichlet or Neumann boundary condition for $g_2(\mathbf{r},\mathbf{r}')$ on S. In this case, explain why $g_2(\mathbf{r},\mathbf{r}')$ is only defined in V_2, and the lower part of (8.1.11) does not hold anymore.

8.3 (a) For an unbounded homogeneous-medium dyadic Green's function, show that $\nabla\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}') = \nabla\times\overline{\mathbf{I}}\,g(\mathbf{r},\mathbf{r}') = -\nabla'\times\overline{\mathbf{I}}\,g(\mathbf{r},\mathbf{r}')$. Hence, show that

$$\left[\nabla\times\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\right]^t = -\left(\nabla'\times\overline{\mathbf{I}}\right)^t g(\mathbf{r},\mathbf{r}') = \nabla'\times\overline{\mathbf{I}}\,g(\mathbf{r},\mathbf{r}') = \nabla'\times\overline{\mathbf{G}}(\mathbf{r}',\mathbf{r}).$$

(b) Derive Equation (8.1.27) in a manner similar to deriving (8.1.26).

(c) Show that $\overline{\mathbf{G}}_2(\mathbf{r},\mathbf{r}')$ need not satisfy the radiation condition in this case.

(d) Show that $\overline{\mathbf{G}}_1(\mathbf{r},\mathbf{r}')$ in (8.1.28) need not be the homogeneous-medium Green's function.

8.4 (a) Derive the identity in Equation (8.1.37).

(b) Derive Equation (8.1.41) and hence, the integral equations in (8.1.42a) and (8.1.42b).

8.5 Show that the z components of the electromagnetic field, after Fourier transforming according to (8.1.43) to (8.1.45), satisfy (8.1.46).

8.6 Derive Equations (8.1.53) and (8.1.54). Show that $G_1(\boldsymbol{\rho},\boldsymbol{\rho}')$ and $G_2(\boldsymbol{\rho},\boldsymbol{\rho}')$ need not be of the form given by (8.1.51). What other forms can you think of?

8.7 Derive the relations (8.1.60) and (8.1.61) and hence, (8.1.62).

8.8 Because of the singularity contained in $\hat{n}\cdot\nabla g(\mathbf{r},\mathbf{r}')$, Equations (8.1.10) and (8.1.11) are undefined when $\mathbf{r}\in S$. But using the definition of the principal value integral as in (8.2.12), the integrals in (8.1.10) and (8.1.11) can be defined even when $\mathbf{r}\in S$. Show that under such a definition, integral equations similar to (8.1.12a) and (8.1.12b) are

$$\phi_{inc}(\mathbf{r}) = \frac{1}{2}\phi_1(\mathbf{r}) + \fint_S dS'\,\hat{n}'\cdot\left[g_1(\mathbf{r},\mathbf{r}')\nabla'\phi_1(\mathbf{r}') - \phi_1(\mathbf{r}')\nabla' g_1(\mathbf{r},\mathbf{r}')\right], \qquad \mathbf{r}\in S,$$

$$0 = \frac{1}{2}\phi_2(\mathbf{r}) - \fint_S dS'\,\hat{n}'\cdot\left[g_2(\mathbf{r},\mathbf{r}')\nabla'\phi_2(\mathbf{r}') - \phi_2(\mathbf{r}')\nabla' g_2(\mathbf{r},\mathbf{r}')\right], \qquad \mathbf{r}\in S.$$

8.9 In (8.2.1), show that $\mathcal{L}_{11}$ and $\mathcal{L}_{21}$ are symmetric operators while $\mathcal{L}_{12}$ and $\mathcal{L}_{22}$ are skew-symmetric operators. Hence, show that their matrix representations by Galerkin's method yield symmetric or skew-symmetric matrices.

8.10 (a) Show that Equation (8.2.14) is always uniquely defined, i.e., it is not discontinuous across the surface S.

(b) Derive Equation (8.2.19) similar to the procedure given in (8.2.9) to (8.2.12).

8.11 (a) Show that for the boundary-element method, the tangential magnetic field on the n-th triangular patch can be expanded as

$$\hat{n}\times\mathbf{H}_n = \sum_{i=1}^{3}\left[h_{i1n}\mathbf{N}_{i1n} + h_{i2n}\mathbf{N}_{i2n}\right] = \sum_{i=1}^{3}\left[\mathbf{N}_{i1n},\mathbf{N}_{i2n}\right]\begin{bmatrix} h_{i1n} \\ h_{i2n}\end{bmatrix}$$
$$= \sum_{i=1}^{3}\overline{\mathbf{N}}_{in}\cdot\mathbf{h}_{in} = \overline{\mathbf{N}}_n^t\cdot\mathbf{h}_n,$$

where

$$\overline{\mathbf{N}}_n^t = \left[\overline{\mathbf{N}}_{1n},\overline{\mathbf{N}}_{2n},\overline{\mathbf{N}}_{3n}\right], \quad \mathbf{h}_n^t = \left[\mathbf{h}_{1n}^t,\mathbf{h}_{2n}^t,\mathbf{h}_{3n}^t\right],$$

and

$$\overline{\mathbf{N}}_{in} = \left[\mathbf{N}_{i1n},\mathbf{N}_{i2n}\right], \quad \mathbf{h}_{in}^t = \left[h_{i1n},h_{i2n}\right].$$

$\mathbf{N}_{i1n}$ and $\mathbf{N}_{i2n}$ are as defined in (8.2.23) for the n-th patch. Hence, $\mathbf{h}_n$ is a column vector of length 6.

(b) If a surface S is approximated by a union of N triangular patches, show that the tangential component of the magnetic field can be expanded as

$$\hat{n}\times\mathbf{H}=\sum_{n=1}^{N}\overline{\mathbf{N}}_n^t\cdot\mathbf{h}_n=\overline{\mathbf{N}}^t\cdot\mathbf{h},$$

where $\overline{\mathbf{N}}^t=\left[\overline{\mathbf{N}}_1^t,\cdots,\overline{\mathbf{N}}_N^t\right]$, and $\mathbf{h}^t=[\mathbf{h}_1^t,\cdots,\mathbf{h}_N^t]$. Hence, $\mathbf{h}$ is a column vector of length $6N$.

(c) The elements of $\mathbf{h}$ consist of the normal components of the edge currents at the nodes of the union of triangular patches that approximates S. Since the normal components are continuous from one edge to another, many of the unknowns in $\mathbf{h}$ are redundant. Hence, the actual number of unknowns needed to approximate $\hat{n}\times\mathbf{H}$ is less than $6N$. Convince yourself that the actual number of unknowns is $2M$ where M is the total number of edges on S and that $2M<6N$.

(d) Given that $\boldsymbol{\eta}$ is a column vector of length $2M$ containing the fundamental unknowns, show that a mapping matrix can be constructed such that

$$\mathbf{h}=\overline{\mathbf{M}}^t\cdot\boldsymbol{\eta},$$

where $\overline{\mathbf{M}}^t$ is a $6N\times 2M$ matrix. Hence, show that

$$\hat{n}\times\mathbf{H}=\overline{\mathbf{N}}^t\cdot\overline{\mathbf{M}}^t\cdot\boldsymbol{\eta}.$$

(e) Given an integral operator $\int_S dS'\,\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\hat{n}\times\mathbf{H}(\mathbf{r}')$, $\mathbf{r}\in S$, show that its matrix representation using Galerkin's method is given by

$$\overline{\mathbf{M}}\cdot\left\langle\overline{\mathbf{N}},\int_S dS'\,\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\overline{\mathbf{N}}^t\right\rangle\cdot\overline{\mathbf{M}}^t,$$

where the inner product is a surface integral on S. What is the dimension of this matrix representation of the integral operator?

8.12 Give an example of $\overline{\mathbf{g}}_n$ in (8.2.27) which is not of a diagonal form, but yet forms a complete set.

8.13 The scattering of a plane wave by a metallic circular cylinder can be solved in closed form:

(a) Using the integral representation of Bessel functions in Chapter 2, Equation (2.2.17), show that a plane wave can be expanded as

$$e^{-ikx}=e^{-ik\rho\cos\phi}=\sum_{n=-\infty}^{\infty}J_n(k\rho)e^{in\phi-in\pi/2}.$$

(b) In two dimensions, the Green's function is [see Equation (3.3.2)]

$$g(\boldsymbol{\rho}-\boldsymbol{\rho}')=\frac{i}{4}H_0^{(1)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|)=\frac{i}{4}\sum_{n=-\infty}^{\infty}J_n(k\rho_<)H_n^{(1)}(k\rho_>)e^{in(\phi-\phi')}.$$

Assuming $\phi_{inc}(\mathbf{r})$ is a plane wave as given above, show that (8.2.31) simplifies to

$$\sum_{n=-\infty}^{\infty}J_n(k\rho)e^{in\phi-in\frac{\pi}{2}}$$
$$=\frac{i}{4}\sum_{n=-\infty}^{\infty}J_n(k\rho)e^{in\phi}H_n^{(1)}(ka)a\int_0^{2\pi}d\phi'\,e^{-in\phi'}\hat{n}'\cdot\nabla'\phi_1(\mathbf{r}'),\quad\rho<a$$

for a plane wave incident on a circular metallic cylinder of radius a.

(c) Find the matrix representation of the integral operator above by expanding $\hat{n}\cdot\nabla'\phi_1(\mathbf{r}')=\sum_{m=-M}^{M}a_m e^{im\phi'}$ and testing with $e^{-ip\phi}$ at $\rho=a$. Show that the simplified version of (8.2.31) in the above reduces to

$$J_p(ka)e^{-ip\frac{\pi}{2}}=\frac{i}{4}J_p(ka)H_p^{(1)}(ka)2\pi a_p.$$

Hence, the matrix representation of the integral operator in (8.2.31) is diagonal in this case.

(d) Show that at the internal resonance of the circular cylinder, both the left-hand side and the right-hand side of the above is zero, rendering a_p undefined.

(e) Show that the resonant sources of (8.2.32) generate no field outside the scatterer. Hence, prove Equation (8.2.33) from the reciprocity theorem (see Chapter 1, Exercise 1.13) for the scalar wave equation. The left-hand side of the equation in (c) vanishes at the internal resonance of the cylinder. Show that (8.2.33) is automatically satisfied for this case.

8.14 A consequence of (8.2.31), by operating on it with $\hat{n}\cdot\nabla$, is that

$$\hat{n}\cdot\nabla\phi_{inc}(\mathbf{r})=\int_S dS'\,\hat{n}\cdot\nabla g_1(\mathbf{r},\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_1(\mathbf{r}'),\quad\mathbf{r}\in S.$$

A combined field integral equation is defined as

$$\phi_{inc}(\mathbf{r})+\lambda\hat{n}\cdot\nabla\phi_{inc}=\int_S dS'\,g_1(\mathbf{r},\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_1(\mathbf{r}')$$
$$+\lambda\int_S dS'\,\hat{n}\cdot\nabla g_1(\mathbf{r},\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_1(\mathbf{r}'),\quad\mathbf{r}\in S.$$

Going through the special case as illustrated in Exercise 8.13, show that the indeterminacy in Exercise 8.13(d) due to internal resonances does not exist for this integral equation if λ is complex.

8.15 For the integral equation in Exercise 8.13(b), we expand

$$\hat{n}\cdot\nabla'\phi_1(\mathbf{r}') = \sum_{m=-M}^{M} a_m e^{im\phi},$$

and test it with $\delta(\rho-a)e^{-ip\phi}$ and $\delta(\rho-a+\Delta)e^{-ip\phi}$ where $p=-M,\cdots,M$ and $\Delta \ll a$. In other words, the integral equation is tested with points on S as well as on points slightly interior to S. Show that the integral equation now reduces to

$$J_p(ka)e^{-ip\frac{\pi}{2}} = \frac{i}{4}J_p(ka)H_p^{(1)}(ka)2\pi a_p, \qquad p=-M,\cdots,M,$$

$$J_p[k(a-\Delta)]e^{-ip\frac{\pi}{2}} = \frac{i}{4}J_p[k(a-\Delta)]H_p^{(1)}(ka)2\pi a_p, \qquad p=-M,\cdots,M.$$

The above is a set of overdetermined equations with $4M$ equations but only $2M$ unknowns. Show that the least-square solution of these overdetermined equations eliminates the internal resonance problem encountered in Exercise 8.13, except at the resonance of the annular region bounded by a and $a-\Delta$, which has very high resonant frequencies.

8.16 By using the addition theorem [see Chapter 3, Equation (3.3.2) and Equation (3.7.4)], show that the scalar Green's function for a homogeneous medium can be expanded as in (8.3.3) for both two and three dimensions. What is $\psi_n(k,\mathbf{r})$ for each of these cases?

Hint: It is sometimes more expedient to use $\cos m\phi$ and $\sin m\phi$ rather than $e^{im\phi}$ dependence so that in the lossless case, "regular part of" is the same as "real part of."

8.17 (a) Show that a closed cavity with the boundary condition $\phi_2(\mathbf{r}')=0$ on S and filled with a material with wavenumber k_2 has a field that satisfies the integral equation

$$0 = \int_S dS'\, g_2(\mathbf{r}-\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_2(\mathbf{r}'), \quad \mathbf{r}\in V_1,$$

where S and V_1 are the same as that in Figure 8.3.1. Now, using (8.3.3) in the above, show that

$$0 = \int_S dS'\, \Re e\psi_n(k_2,\mathbf{r}')\,\hat{n}'\cdot\nabla'\phi_2(\mathbf{r}'), \quad \text{for all } n.$$

The above integral equation has only a trivial solution for $\hat{n}'\cdot\nabla'\phi_2(\mathbf{r}')$ except at the resonant frequencies of the cavity. This implies that at the nonresonant frequencies of the cavity, the only function that is orthogonal to $\Re e\psi_n(k_2,\mathbf{r}')$ for all n is zero. Consequently, $\Re e\psi_n(k_2,\mathbf{r}')$ is complete on S, except at the resonant frequencies of the cavity.

(b) Similarly, prove that $\hat{n}\cdot\nabla'\Re e\psi_n(k_2,\mathbf{r}')$ is complete on S except at the resonant frequencies of the cavity formed by S filled with a material with wavenumber k_2. What is the boundary condition on the wall of this cavity? Could any of the resonant frequencies in case (a) coincide with the resonant frequencies in this case?

8.18 Derive Equation (8.3.13) from (8.3.12).

8.19 (a) Derive the equivalence of (8.3.15) for an impenetrable scatterer with a homogeneous Dirichlet boundary condition on S. Give the definition for Q_{nm} in this case.

(b) Show that at the internal resonances of the cavity formed by S filled with a material with wavenumber k_2, the matrix Q_{nm} is ill-conditioned.

(c) Show that this problem can be remedied by using a linearly independent set to expand the surface field rather than those suggested by (8.3.9) and (8.3.10).

(d) Explain why the internal resonances do pose a problem for the $\overline{\mathbf{Q}}$ matrix of a penetrable scatterer given by (8.3.16).

8.20 Derive the equivalence of (8.3.7) for a circular metallic cylinder. Show that the internal resonance problem encountered in Exercise 8.13 does not exist here.

8.21 Show that in spherical coordinates, the dyadic Green's function can be expressed as (8.3.19). Identify the vector wave functions (see Chapter 7).

8.22 Derive Equations (8.3.21) and (8.3.22) from Equations (8.3.17) and (8.3.18).

8.23 Show that the set $\hat{n}\times\Re e\psi_m(k_2,\mathbf{r}')$ or $\hat{n}\times\nabla'\times\Re e\psi_m(k_2,\mathbf{r}')$ is complete on the surface S except at the internal resonance of the cavity formed by S and filled with a material with wavenumber k_2.

8.24 Prove the identity (8.3.27).

8.25 From the reciprocity theorem for the scalar wave equation, prove that $\overline{\mathbf{T}}$, and hence $\overline{\mathbf{S}}$, are symmetric matrices.

8.26 (a) From the scalar wave equation $(\nabla^2+k^2)\phi=0$, derive the energy conservation theorem that $\int_S dS\,\hat{n}\cdot(\phi\nabla\phi^*-\phi^*\nabla\phi)=0$ when k is real. Hence, $\mathbf{F}=\phi\nabla\phi^*-\phi^*\nabla\phi$ is an energy flux.

(b) Show that for a lossless medium,

$$\psi_n(-k_1, \mathbf{r}) = \psi_n^*(k_1, \mathbf{r}),$$

where $\psi_n(k_1, \mathbf{r})$ is derived in Exercise 8.16.

(c) Show that $\int_S dS\,\hat{n}\cdot[\psi\nabla\psi^t - (\nabla\psi)\psi^t] = 0$ when S is a closed surface, and ψ is as defined for (8.4.8).

Hint: Derive it first for when S is a circle or a sphere, and deform it to an arbitrary S later. The deformation is allowed because $\nabla\cdot[\psi\nabla\psi^t - (\nabla\psi)\psi^t] = 0$ in a region excluding the origin.

(d) Using (8.4.12), (b), and (c), show that

$$\int_S dS\,\hat{n}\cdot(\phi\nabla\phi^* - \phi^*\nabla\phi) = \frac{1}{4}\mathbf{a}^t\cdot\int_S dS\,\Big[\psi^*\nabla\psi^t - (\nabla\psi^*)\psi^t + \overline{\mathbf{S}}^t\cdot(\psi\nabla\psi^\dagger - (\nabla\psi)\psi^\dagger)\cdot\overline{\mathbf{S}}^*\Big]\cdot\mathbf{a}^*.$$

(e) Show that

$$\int_S dS\,[\psi^*\nabla\psi^t - (\nabla\psi^*)\psi^t] = ic\overline{\mathbf{I}},$$

where c is a real constant.

Hint: The Wronskian of Bessel functions may be useful here.

(f) Therefore, show that in order for the integral in (d) to vanish to conserve energy, $\overline{\mathbf{S}}\cdot\overline{\mathbf{S}}^\dagger = \overline{\mathbf{S}}\cdot\overline{\mathbf{S}}^* = \overline{\mathbf{I}}$.

8.27 (a) Using Rayleigh's hypothesis method, derive the $\overline{\mathbf{T}}$ matrix for an impenetrable scatterer with a homogeneous Dirichlet boundary condition. Show that this is the transpose of the $\overline{\mathbf{T}}$ matrix derived by the extended-boundary-condition method.

(b) Explain why the error in Rayleigh's hypothesis method and the EBC method should be of the same order if the same number of terms are used in both methods.

8.28 (a) Derive the $\overline{\alpha}_{ji}$ and $\overline{\beta}_{ji}$ matrices in cylindrical coordinates. Show that $\overline{\beta}_{ji} = \Re e\overline{\alpha}_{ji}$, where "$\Re e$" stands for "the regular part of."

(b) Derive the $\overline{\alpha}_{ji}$ and $\overline{\beta}_{ji}$ matrices in spherical coordinates. Show that $\overline{\beta}_{ji} = \Re e\overline{\alpha}_{ji}$ (also see Appendix D).

8.29 Count the number of matrix multiplications required in (8.6.17) and (8.6.18) at each iteration, and that required after N iterations for N scatterers. Show that the number of matrix multiplication is proportional to N^2 when $N \to \infty$.

8.30 Using the EBC method, derive the $\overline{\mathbf{R}}_{10}$ and $\overline{\mathbf{T}}_{10}$ matrices defined in Equations (8.7.7a) and (8.7.7b) for the geometry shown in Figure 8.7.2.

8.31 (a) By expanding (8.7.23) and (8.7.28) into a geometric series, give a physical explanation for each term of the series.

(b) Derive the relationships (8.7.25) and (8.7.26).

8.32 Derive the matrix representation of Equation (8.8.3); hence, derive (8.8.8) and (8.8.9).

8.33 (a) Explain why $\overline{\mathbf{L}}$, and hence $\overline{\mathbf{M}}^{-1}$ in (8.8.15), are singular at the resonant frequencies of the cavity formed by S. Show that for a time harmonic solution, this poses a problem only for a lossless medium filling the cavity.

(b) Show that for a finite vector $\mathbf{d}$, $\mathbf{c}$ is infinite at the resonant frequencies in (8.8.15). At the resonant frequencies of the cavity, however, $\hat{n}\cdot\nabla\phi_1(\mathbf{r}) = 0$, and hence, $\mathbf{d} = 0$ from (8.8.12b). Therefore, $\mathbf{c}$ must be finite while $\mathbf{d} = 0$ at resonances.

(c) At the resonances of the cavity, $\overline{\mathbf{L}}$, which is symmetric, has zero eigenvalues. Using the singular value decomposition method, show that $\overline{\mathbf{L}} = \overline{\mathbf{S}}^t\cdot\overline{\lambda}\cdot\overline{\mathbf{S}}$ where $\overline{\lambda}$ is a diagonal matrix containing the eigenvalues of $\overline{\mathbf{L}}$. Some of these eigenvalues are zero at the resonances of the cavity.

(d) Because of the zero eigenvalues of $\overline{\mathbf{L}}$, its inverse $\overline{\mathbf{L}}^{-1}$ has infinite eigenvalues. Show that $\overline{\mathbf{M}}$ also has infinite eigenvalues at the resonances of the cavity. Hence, $\overline{\mathbf{M}}$ is not computable when $\overline{\mathbf{L}}$ is singular.

(e) Show that by setting the zero eigenvalues of $\overline{\mathbf{L}}$ to a small, nonzero number, $\overline{\mathbf{M}}$ is computable. Instead of having infinite eigenvalues, $\overline{\mathbf{M}}$ has large eigenvalues. In this manner, $\overline{\mathbf{M}}^{-1}$ in (8.8.22) can be found to a degree of accuracy permitted by machine precision.

8.34 A less computationally intensive method of finding $\overline{\mathbf{L}}^{-1}$ to within machine precision is to use the regularization method.

(a) Explain why the internal resonance poses a problem only for lossless media in S. Show that the eigenvalues of $\overline{\mathbf{L}}$ are always real in this case.

(b) If a new $\overline{\overline{\mathbf{L}}}$ is defined such that $\overline{\overline{\mathbf{L}}} = \overline{\mathbf{L}} + i\delta\overline{\mathbf{I}}$, where $i\delta$ is a pure imaginary number, show that $\overline{\overline{\mathbf{L}}}$'s eigenvalues will never be zero. $i\delta$ can be chosen just large enough so that $\overline{\overline{\mathbf{L}}}^{-1}$ can be found without overflowing the computer floating-point capability. In this manner, $\overline{\mathbf{M}}$ can be computed, as can $\overline{\mathbf{M}}^{-1}$.

Alternatively, the resonance problem can be alleviated by assuming a small loss in V_1.

8.35 (a) Assuming that an impedance boundary condition such that $\hat{n} \cdot \nabla g_1 = Z g_1$ on S is imposed, formulate the scattering problem by an inhomogeneous body using the finite-element method together with this impedance boundary condition.

(b) Explain why the internal resonance problem would not exist in this case.

8.36 The field in V_1 in Equation (8.8.6) can be decomposed as $\phi_1 = \phi_{inc} + \phi_{sca}$, where ϕ_{sca} is the scattered field due to the inhomogeneity.

(a) Show that if $g_1(\mathbf{r}, \mathbf{r}')$ in (8.8.6) satisfies (8.8.3) plus the radiation condition at infinity, then

$$\int_S dS' \left[g_1(\mathbf{r}, \mathbf{r}')\, \hat{n}' \cdot \nabla' \phi_{sca}(\mathbf{r}') - \phi_{sca}(\mathbf{r}')\, \hat{n}' \cdot \nabla' g_1(\mathbf{r}, \mathbf{r}')\right] = 0, \quad \mathbf{r} \in V_1$$

in Figure 8.8.1. Hence, Equation (8.8.6) can be written as

$$\phi_1(\mathbf{r}) = \int_S dS' \left[g_1(\mathbf{r}, \mathbf{r}')\, \hat{n}' \cdot \nabla' \phi_{inc}(\mathbf{r}') - \phi_{inc}(\mathbf{r}')\, \hat{n}' \cdot \nabla' g_1(\mathbf{r}, \mathbf{r}')\right], \quad \mathbf{r} \in V_1.$$

(b) The above shows that $\phi_1(\mathbf{r})$ inside V_1 is known once the requisite $g_1(\mathbf{r}, \mathbf{r}')$ satisfying the radiation condition is known. Moreover, once $\phi_1(\mathbf{r})$ is known, then ϕ_{sca} is known inside V_1 and on S since $\phi_1(\mathbf{r}) = \phi_{inc}(\mathbf{r}) + \phi_{sca}(\mathbf{r})$. But in (8.8.5), $\phi_0 = \phi_{inc} + \phi_{sca}$ in V_0. Hence, show that if the homogeneous-medium Green's function $g_0(\mathbf{r}, \mathbf{r}')$ satisfies the radiation condition at infinity, then

$$\int_S dS' \left[g_0(\mathbf{r}, \mathbf{r}')\, \hat{n}' \cdot \nabla' \phi_{inc}(\mathbf{r}') - \phi_{inc}(\mathbf{r}')\, \hat{n}' \cdot \nabla' g_0(\mathbf{r}, \mathbf{r}')\right] = 0, \quad \mathbf{r} \in V_0.$$

Hence, from (8.8.5), deduce that

$$\phi_{sca} = -\int_S dS' \left[g_0(\mathbf{r}, \mathbf{r}')\, \hat{n}' \cdot \nabla' \phi_{sca}(\mathbf{r}') - \phi_{sca}(\mathbf{r}')\, \hat{n}' \cdot \nabla' g_0(\mathbf{r}, \mathbf{r}')\right], \quad \mathbf{r} \in V_0.$$

In conclusion, if $g_1(\mathbf{r}, \mathbf{r}')$ is known inside V_1 and on S, then the field everywhere could be found. But the requisite $g_1(\mathbf{r}, \mathbf{r}')$ satisfying the radiation condition can be found approximately using an absorbing boundary condition on S. An absorbing boundary condition may be of the form $\hat{n} \cdot \nabla g_1 = Z g_1$ on S, so that all outgoing waves are absorbed on S emulating the radiation condition. Such a manner of formulating an FEM problem with this boundary condition is described in Exercise 8.35. Moreover, absorbing boundary conditions are described in Chapter 4.

8.37 Explain why the second term in (8.9.13) is related to magnetic polarization charges.

8.38 Derive Equation (8.9.24) from Equation (8.9.23) and show that N_{mn} is a symmetric matrix. Explain why the matrix representation of the integral equation is symmetrized by a multiplication by $O(\mathbf{r})$ as in (8.9.19).

8.39 The Neumann series expansion of an integral equation is the higher-dimensional analogue of the Taylor's series expansion of a scalar function. By using the $k^2 - k_b^2$ as a small parameter, show that the error in (8.10.2) is of higher order.

8.40 (a) Assume an integral of the form

$$I = \int_V d\mathbf{r}'\, g(\mathbf{r}, \mathbf{r}') q(\mathbf{r}'),$$

where $g(\mathbf{r}, \mathbf{r}') = e^{ik_b|\mathbf{r}-\mathbf{r}'|}/4\pi|\mathbf{r} - \mathbf{r}'|$. By letting $\boldsymbol{\eta} = \mathbf{r}/L$ and $\boldsymbol{\eta}' = \mathbf{r}'/L$, where L is the typical size of the volume V (i.e., $V \simeq L^3$), show that

$$I = L^2 \int_{V/L^3} d\boldsymbol{\eta}' \frac{e^{ik_b L|\boldsymbol{\eta}-\boldsymbol{\eta}'|}}{4\pi|\boldsymbol{\eta} - \boldsymbol{\eta}'|} q(\boldsymbol{\eta}' L).$$

The integral now is mainly dimensionless except for the dimension of q. If $k_b L \ll 1$ or $L \ll \lambda_b$, then show that

$$I \simeq L^2 \int_{V/L^3} d\boldsymbol{\eta}' \frac{q(\boldsymbol{\eta}' L)}{4\pi|\boldsymbol{\eta} - \boldsymbol{\eta}'|} \sim O(L^2)\tilde{q}(\boldsymbol{\eta}' L).$$

When should $\tilde{q}$ be of the same order as q? If so, then I is $O(L^2 q)$.

(b) Show that the above can be obtained more quickly by dimensional analysis, i.e., by assuming that $g(\mathbf{r}, \mathbf{r}') \sim 1/L$, $\int d\mathbf{r}' \sim L^3$.

8.41 (a) For a plane wave at normal incidence on a dielectric slab of thickness L, find the exact solution of the reflected wave.

(b) Derive the approximation of the reflected wave when $\frac{\epsilon}{\epsilon_b} - 1 \to 0$, where ϵ is the permittivity of the dielectric slab and ϵ_b is the permittivity of the background.

(c) Derive the reflected wave using the Born approximation, and show that this result reduces to that in (b) only if (8.10.10) is satisfied.

8.42 (a) For a scalar wave equation, show that (8.10.11) is the constraint for the validity of the Born approximation at low frequencies.

(b) Show that the corresponding constraint for two dimensions is

$$k_b^2 L^2 \ln(kL) \Delta\epsilon_r \ll 1,$$

and that for one dimension is $k_b L \Delta\epsilon_r \ll 1$.

8.43 (a) For a homogeneous dielectric slab of thickness L with a wave normally incident on it, derive the exact solution for the reflected wave as well as the wave inside the slab using the method of Chapter 2.

(b) Derive the field inside the slab using the Rytov approximation. Show that this result reduces to that of (a) inside the slab when (8.10.33) is satisfied but not outside the slab.

References for Chapter 8

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